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Permeable Rings and Their Extensions

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

Let us call a ring R to be right permeable if for any $a \in R$; $Ra = 0$, then $aR = 0$. Left permeable and permeable rings are defined analogously. These rings are generalized reversible rings with a privileging role that permeability inherited in its several extensions where reversibility seized to be inherited. It will be proved that full matrix ring, polynomial ring, Laurent polynomial ring, Dorroh extension, group ring and Ore extensions of a right (left) permeable ring are right (left) permeable rings. Moreover, the same holds for Barnett matrix rings with their extensions in different quotient polynomials and matrix forms.

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1 Introduction

The class of reversible rings is the most popular class of rings which commute over zero. However, one may notice that, like commutativity, these rings are no longer reversible over several extensions, including full matrix and polynomial extensions. Here we introduce a class of rings which is a generalization of reversible rings and inherits its basic property in several extensions. We call such rings permeable and define that:

A ring R is *right permeable* (*left permeable*) if $Ra = 0 \Rightarrow aR = 0$ ($aR = 0 \Rightarrow Ra = 0$). A ring is *permeable* if it is both right and left permeable. A *Proper right permeable* ring is one which is right permeable but not left permeable, same for a proper left permeable. A ring is *trivially right (left) permeable* if 0 is the only element which satisfies the condition of right (left) permeability.

In this note we will demonstrate that: full matrix rings, polynomial rings, Laurent, skew and Ore extensions of a right (left) permeable ring are right (left) permeable. Same holds for Dorroh and group ring extensions. Moreover, it also holds for Barnett matrix rings with their extensions in different quotient polynomials and matrix forms.

Throughout this note all rings are assumed to be associative with or without the multiplicative identity (in short we will use the term identity or unity). If a ring has the identity we will specifically mention it. According to Cohn [1] a ring R is *reversible* if for any $a, b \in R$, $ab = 0$ then $ba = 0$, while Lambek in [2] defined that a ring R is *symmetric* if for any triple $a_1, a_2, a_3 \in R$, $a_1a_2a_3 = 0$, then $a_{s(1)}a_{s(2)}a_{s(3)} = 0$, where $s \in S_3$, and S_3 is the symmetric group on $\{1, 2, 3\}$. These two classes of rings are in fact useful generalizations of commutativity at zero and have extensively been studied in the last two decades.

Most of the studies of reversible and symmetric rings and their generalizations are focusing on the elements of the ring but not on the ring itself. While for permeable rings, where the term is deduced from permeability, the elements which annihilate the ring commute with the ring (or passes through the ring). If R is a ring with identity, then R is trivially permeable, while if R has a left (right) identity, then R is trivially right (left) permeable. Moreover, if R is reversible, then it is permeable. So the class of permeable rings contains at least the classes of reversible rings and all rings with identity. Reduced rings (rings without non-zero nilpotent elements) and the zero rings ($R^2 = 0$) are also permeable.

In general, symmetric rings are not reversible. For instance, the ring of strictly upper triangular matrices of order three defined over any non-zero ring is symmetric but not reversible. Such rings are also not permeable. Anderson and Camillo in [3] defined ZC_n rings, where a ring R satisfies the condition ZC_n : if for $n \geq 2$ and $a_1a_2 \dots a_n = 0 \Rightarrow a_{s(1)}a_{s(2)} \dots a_{s(n)} = 0$, $a_i \in R$, $s \in S_n$, where S_n is the group of permutations on n elements. In [4], Definition 2.1., a ring R is termed as *symmetric* if R satisfies ZC_n for all $n \geq 2$. This definition of symmetric rings appeared to be stronger than the original definition introduced by Lambek in [2] and implies reversibility for rings with or without identity. But several classes of rings which are symmetric under the classical definition of Lambek are not symmetric under the definition given in [4]. So, in this note, we prefer to call a ring that satisfies ZC_n for all $n \geq 2$ a *fully symmetric ring.* One may redefine that a ring is fully symmetric if it is symmetric and reversible. These rings are clearly permeable.

The crisis of identity in the study of symmetric rings becomes more visible when non-commutative Klien-4 rings and their various extensions are studied (e.g. see [5, 6]). A ring R is right (respt. left) symmetric, as defined in [6] if $a, b, c \in R$, such that $abc = 0$ implies that $acb = 0$ (respt. $bac = 0$). Clearly, the Klien-4 ring $\mathcal{V}_4$ as defined in Example 5.3 is right symmetric and is not symmetric. So one concludes that

$$Fully\ Symmetric \Longrightarrow Symmetric \Longrightarrow Right\ (left)\ Symmetric$$

and these implications are irreversible in general. However, for rings with 1, all these three different notions coincide. It is interesting to note the following:

Fully symmetric rings are permeable.
Symmetric rings, in general, are not permeable.

First statement is due to the fact that a fully symmetric ring is reversible, second is clear from Example 3.2.1.

Some other generalizations of commutativity at zero which have link with the symmetric and reversible rings, are semi-commutative and McCoy: A ring R is *semi-commutative* (or SI, IFP, etc.), termed mainly, after Bell [7], if for $a, b \in R$ with $ab = 0$, then $aRb = 0$. R is said to be *right McCoy* (respectively *left McCoy*), as introduced by Nielsen in [8], if for each pair of non-zero polynomials $f(x), g(x) \in R[x]$ with $f(x)g(x) = 0$, then there exists a non-zero element $r \in R$ with $f(x)r = 0$ (respectively $rg(x) = 0$). A ring is *McCoy* if it is both left and right McCoy.

The right (left) annihilator of a ring R is $ann_r(R) = \{a \in R : Ra = 0\}$ ($ann_l(R) = \{a \in R : aR = 0\}$). As usual, we will denote the center of a ring R by $Cent(R)$, $J(R)$ is its Jacobson radical, $P(R)$ is the prime radical, and $N(R)$ is the set of all nilpotent elements of R. $\mathbb{Z}_n$ is the set of integers modulo n.

The following chart shows all implications among these types of rings. Of course, these implications are reflexive, anti-symmetric and transitive, but not symmetric.

$Commutative$	$\Longrightarrow$	$Fully$ $Symmetric$	$\Longrightarrow$	$Reversible$	$\Longrightarrow$	$Permeable$
		$\Uparrow$		$\Downarrow$		$\Downarrow^*$
		$Reduced$		$Semi-$ $Commutative$		$McCoy$

The implications regarding permeable rings are straightforward (except the implication $*$ which is also straightforward but holds for non-trivially permeable rings). For other implications with examples and counter examples, we refer to [4, 8, 9]. The above directed diagram will be referred to as the "Chart".

In Section 2 we have given several examples and studied some properties of (right, left) permeable rings and simultaneously we compare them with other classes of rings to fill the missing steps of the Chart and other statements presented in this section. In general, the complete matrix rings are failed to inherit several properties of the base rings, except a few lucky rings, these are the Morita invariant rings (in particular for rings with 1). Permeable rings are included among such lucky rings, this is the first statement of Section 3. Then several other classes of matrix rings and subrings are studied in this section, several of them are permeable, several do not. Permeable rings can conveniently be extended to various polynomial rings and also for Ore extension rings. These are proved in Section 4. In this last section some extensions in the form of Barnett matrix rings and factor polynomial rings are studied and further extended. Permeable power series are introduced along with some properties and a counter example is posed at the end.

2 Properties and Examples

In this section we characterize permeable rings with respect to annihilators and several examples and counter examples will be presented. The following elementary results are useful and we will use them freely without mentioning them throughout this work. Note that $ann_r(R)$ and $ann_l(R)$ are ideals of R. We redefine that

Definition 2.1. A ring R is right permeable (respt. left permeable) if for any $a \in R$, $Ra = 0$ then $aR = 0$ (respt. $aR = 0$ then $Ra = 0$), permeable if it is both right and left permeable. We say that R is proper right permeable if it is right permeable but not left permeable, same for proper left permeable, while a ring is trivially right (left) permeable if 0 is the only element which satisfies the condition of right (left) permeable.

Lemma 2.2. *Let R be a ring. Then the following are equivalent:*

1. R is right permeable.
2. $ann_r(R) \subseteq Cent(R)$.
3. $ann_r(R) \subseteq ann_l(R)$.

Proof. (1) $\Rightarrow$ (2) Let R be right permeable and $c \in ann_r(R)$. Then $rc = 0 \Longrightarrow cr = 0\ \forall r \in R$ and so $c \in Cent(R)$.
(2) $\Rightarrow$ (3) Let $c \in ann_r(R)$. Since $rc = 0 = cr\ \forall r \in R$, then $c \in ann_l(R)$.
(3) $\Rightarrow$ (1) Let $Rc = 0$ for some $c \in R$. Then $c \in ann_r(R) \subseteq ann_l(R)$ and so $cR = 0$. □

Corollary 2.3. *Let R be a ring. Then the following are equivalent:*

1. *R is permeable.*
2. *$ann_r(R) \cup ann_l(R) \subseteq Cent(R)$.*
3. *$ann_r(R) = ann_l(R)$.*

Lemma 2.4. *Let R be a proper right permeable ring and let S be a non-trivially permeable ring. Then $R \times S$ is a non-trivially proper right permeable ring.*

Proof. By Theorem 2.8, $R \times S$ is right permeable. Since S is non-trivially permeable, then there exists $0 \neq a \in S$ such that $aS = Sa = 0$ and so $(0, a)R \times S = R \times S(0, a) = (0, 0)$. On the other hand there is $0 \neq b \in R$ such that $bR = 0 \neq Rb$ since R is proper right permeable, implying $(b, 0)R \times S = 0 \neq R \times S(b, 0)$. Therefore $R \times S$ is a non-trivially proper right permeable ring. □

Example 2.5.

1. Let $\mathcal{A}_4 = \{0, x, y, z\}$ be the ring such that $(\mathcal{A}_4, +) \cong (\mathbb{Z}_4, +)$, where $x \equiv 1$, $y \equiv 2$ and $z \equiv 3$. Multiplication is defined as $yr = ry = 0\ \forall r \in \mathcal{A}_4$ and $x^2 = xz = zx = z^2 = y$. This ring is commutative and non-trivially permeable.
2. Now we construct an example of a commutative ring without 1 which is non-trivially permeable. In fact it is a generalization of above example. Consider the set:

   $$\mathcal{A}_{2n} = \{0, 1, 2, \ldots, 2n-1\},\ n \geq 2.$$

   Call an element e (respt. d) $\in \mathcal{A}_{2n}$ an *even* (respt. *odd*) *element* if $e \in \{0, 2, 4, \ldots, 2n-2\}$ (respt. $d \in \{1, 3, 5, \ldots, 2n-1\}$). Then $\mathcal{A}_{2n}$ is a commutative ring, where $(\mathcal{A}_{2n}, +) = (\mathbb{Z}_{2n}, +)$ and multiplication is defined by the rules $ea = ae = 0\ \forall a \in \mathcal{A}_{2n}$ and $dd_1 = n$ for all odd elements $d, d_1 \in \mathcal{A}_{2n}$. Let $(x_1, y_1), (x_2, y_2) \in \mathcal{A}_{2n} \times \mathcal{A}_{2n}$ such that $(x_1, y_1) = (x_2, y_2)$. If $x_1 = e$ or $y_1 = e$ where e is even, then $x_2 = e$ or $y_2 = e$ and hence $x_1y_1 = 0 = x_2y_2$. Thus we can assume that x_1 and y_1 to be odd implying $x_1y_1 = n = x_2y_2$. This proves that the multiplication is well defined. For associativity, let $r, s, t \in \mathcal{A}_{2n}$. If one of these elements is even, then $r(st) = (rs)t = 0$. Thus, we suppose that all of them are odd and we distinguish two cases. If n is odd, then $r(st) = rn = n = nt = (rs)t$. So we can assume that n is even, and hence $r(st) = rn = 0 = nt = (rs)t$. We have three cases for distributivity, where e is even, $a_1, a_2 \in \mathcal{A}_{2n}$ and d, d_1, d_2 are odd:

$$case\ 1: e(a_1 + a_2) = 0 = ea_1 + ea_2.$$

$$case\ 2: d(d_1 + d_2) = de = 0 = n + n = dd_1 + dd_2.$$

$$case\ 3: d(e + d_1) = dd_2 = n = 0 + n = de + dd_1.$$

Hence $\mathcal{A}_{2n}$ is a ring, commutative and without 1.

Obviously, it is non-trivially permeable $\forall n \geq 2$, since

$$ann_r(\mathcal{A}_{2n}) = \{0, 2, \ldots, 2n-2\} = ann_l(\mathcal{A}_{2n}).$$

If $n = 1$, then $\mathcal{A}_2$ is the Galois field of order 2, and it is trivially permeable. If $n \geq 2$ is an even integer, then it can be checked easily that the ring $\mathcal{A}_{2n}$ is a Jacobson ring in the sense that its Jacobson radical $J(\mathcal{A}_{2n}) = \mathcal{A}_{2n}$ and also $P(\mathcal{A}_{2n}) = \mathcal{A}_{2n} = N(\mathcal{A}_{2n})$. Moreover, $\mathcal{A}_{2n}$ is nilpotent of index 3 and the factor ring $\mathcal{A}_{2n}/ann_r(\mathcal{A}_{2n})$ is a zero-ring of order two, which is non trivially permeable.

Finally, if $n \geq 3$ is odd, then $J(\mathcal{A}_{2n}) = ann_r(\mathcal{A}_{2n}) = P(\mathcal{A}_{2n})$. Also $\mathcal{A}_{2n}$ is not nilpotent, since $\mathcal{A}_{2n}^m = \{0, n\}$ $\forall m \geq 2$ and that $\mathcal{A}_{2n}/ann_r(\mathcal{A}_{2n}) \cong \mathcal{A}_2$.

3. Let $\mathcal{V}_4 = \{0, a, b, c\}$ be the non-commutative Klein 4-ring with characteristic two and the additive structure of $\mathcal{V}_4$ is isomorphic to the Klein 4-group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Its elements follow the multiplication rules: $a^2 = ab = a$; $ba = b^2 = b$; $c = a + b$. Then $\mathcal{V}_4 c = 0$ but $c\mathcal{V}_4 \neq 0$. Hence $\mathcal{V}_4$ is not right permeable, it is trivially proper left permeable. On the similar ground, $\mathcal{V}_4^{op}$, the opposite ring of $\mathcal{V}_4$, is not left permeable, it is trivially proper right permeable. The $2^m th$ extension of non-commutative Klien 4-ring without 1 is constructed in Theorem 3.2 of [6] and is denoted by $\mathcal{V}_{2^m}$. Assume that $a, b \in \mathcal{V}_{2^m}$ and X is a set of generators, where

$$a = \sum_{i=1}^{\alpha} x_i, b = \sum_{j=1}^{\beta} y_j, \quad x_i, y_j \in X, \alpha, \beta \in \mathbb{Z} \text{ such that } 1 \leq \alpha, \beta \leq m.$$

According to Theorem 3.2 of [6], we have

$$ab = \sum_{i=1}^{\alpha} x_i \sum_{j=1}^{\beta} y_j = \sum_{i=1}^{\alpha} \beta x_i = \begin{cases} 0 & \text{when } \beta \text{ is even} \\ a & \text{when } \beta \text{ is odd} \end{cases}$$

Thus $ann_r(\mathcal{V}_{2^m}) = \{b \in \mathcal{V}_{2^m} : \beta \text{ is even}\}$ where b is a right identity if β is odd and so $ann_l(\mathcal{V}_{2^m}) = \{0\}$. This proves that $\mathcal{V}_{2^m}$ is trivially proper left permeable. Analogously, $\mathcal{V}_{2^n}^{op}$ is trivially proper right permeable.

4. The direct product ring $\mathcal{V}_{2^m} \times \mathcal{A}_{2n}$ is a non-trivial proper left permeable ring, while $\mathcal{V}_{2^m}^{op} \times \mathcal{A}_{2n}$ is a non-trivially proper right permeable ring $\forall m, n \geq 2$, by Lemma 2.4.

5. Now consider the ring $R = \mathcal{V}_4 \times \mathcal{V}_4^{op}$. Since $(0, c)f(x) = 0$ and $f(x)(c, 0) = 0$ for all $f(x) \in R[x]$, the ring $\mathcal{V}_4 \times \mathcal{V}_4^{op}$ is McCoy. Clearly, this ring is not permeable. It is trivial to verify that every non-trivially permeable ring is a McCoy ring. This verifies the last part of the Chart.

Now we introduce a condition for a permeable ring to be fully symmetric.

The $(CA)-$ condition: A ring R satisfies (CA), if $ac = 0$ (or $ca = 0$) with $aR \neq 0$ and $Ra \neq 0$, then $Rc = 0$ or $cR = 0$.

Theorem 2.6. *If R is permeable and satisfying (CA), then R is fully symmetric.*

Proof. Recall that $ann_r(R) = ann_l(R)$ by Corollary 2.3. Let $a_1, a_2, \ldots, a_n \in R$ such that $a_1a_2 \ldots a_n = 0$. Suppose that $a_i \notin ann_r(R)$ for all i. Then $a_{i_1}a_{i_2} \neq 0$ for all $1 \leq i_1, i_2 \leq n$, otherwise $a_{i_1} \in ann_r(R)$ or $a_{i_2} \in ann_r(R)$. Thus $a_{i_1}a_{i_2}a_{i_3} \neq 0$ for all $1 \leq i_1, i_2, i_3 \leq n$. Continuing this way we get $a_{i_1}a_{i_2} \ldots a_{i_n} \neq 0$ for all $1 \leq i_1, i_2, \ldots, i_n \leq n$, a contradiction. Implying that some of the a_i's belongs to $ann_r(R)$ and we are done. □

Example 2.7.

1. Note that $\mathcal{V}_4$ satisfies (CA) but it is not permeable and not fully symmetric (or even symmetric), since $ca = c$ while $ac = 0$.
2. For rings satisfying both conditions, let D be any domain (a ring with no non-zero zero divisors) and R any zero ring. Then $D \times R$ is the desired ring.

Next theorem gives two sided statement about the direct product of permeable rings.

Theorem 2.8. *Let $\{R_i : i \in I\}$ be an indexed family of rings. Then $R = \prod_{i \in I} R_i = \{(a_i)_{i \in I} : a_i \in R_i \ for \ i \in I\}$ (the external direct product of R_i) is right permeable if and only if R_i is right permeable for each $i \in I$.*

Proof. First we show that $ann_r(R) = \prod_{i \in I} ann_r(R_i)$. Assume on the contrary that $(a_i)_{i \in I} \in ann_r(R)$, where $a_j \notin ann_r(R_j)$. Then there exists $b_j \in R_j$ such that $b_ja_j \neq 0$. Thus, for $(b_i)_{i \in I}$ with b_j in component j $(b_i)_{i \in I}(a_i)_{i \in I} \neq 0$, a contradiction. Let R be right permeable and let $(a_i)_{i \in I} \in ann_r(R)$ where $a_j \in ann_r(R_j)$ and 0 elsewhere. Take $(b_i)_{i \in I} \in R$ where $b_j \in R_j$ and 0 elsewhere. Then $(a_i)_{i \in I}(b_i)_{i \in I} = (b_i)_{i \in I}(a_i)_{i \in I} = 0$. Hence $a_jb_j = 0 \; \forall b_j \in R_j, a_j \in ann_r(R_j)$. This proves that R_j is right permeable for all $j \in I$. Conversely, suppose that each R_i is right permeable . Let $(a_i)_{i \in I} \in ann_r(R)$ and $(b_i)_{i \in I} \in R$. Then $(a_i)_{i \in I}(b_i)_{i \in I} = (a_ib_i)_{i \in I} = 0$. □

Corollary 2.9. *Any ring R with identity can be embedded in a non-trivially permeable ring and so R is embeddable in a McCoy ring.*

Dorroh Extension: Let R and S be rings such that $R \subseteq Cent(S)$. Then the ring $D = \{(r, s) : r \in R, s \in S\}$, where the addition is defined component wise and multiplication is defined by the rule $(r_1, s_1)(r_2, s_2) = (r_1r_2, r_1s_2 + r_2s_1 + s_1s_2)$ is called the Dorroh extension of R by S.

Theorem 2.10. *Let R and S be rings such that $R \subseteq Cent(S)$ and $ann_r(R) \subseteq ann_r(S)$. Then the Dorroh extension D of R by S is right permeable if and only if S is right permeable.*

Proof. $ann_r(D) = (ann_r(R), ann_r(S))$. Otherwise, if $(a, b) \in ann_r(D)$, where $a \notin ann_r(R)$ or $b \notin ann_r(S)$, then there exists $a' \in R$ and $b' \in S$ such that $a'a \neq 0$ or $b'b \neq 0$. Thus $(a', 0)(a, b) \neq (0, 0)$, a contradiction, and so $a \in ann_r(R)$. If $b \notin ann_r(S)$, then $(0, b')(a, b) = (0, b'b) \neq (0, 0)$, a contradiction. Let D be right permeable and $s' \in ann_r(S)$. Then $(0, s)(0, s') = (0, ss') = (0, 0) = (0, s')(0, s) = (0, s's) \; \forall s \in S$, which implies $s's = 0 \; \forall s \in S$. Conversely, assume that S is right permeable. Then R is right permeable, since $R \subseteq Cent(S)$ and hence commutative. Let $(r', s') \in ann_r(D)$. Then $r' \in ann_r(R)$ and $s' \in ann_r(S)$ and so $(r', s')(r, s) = (0, 0) \; \forall (r, s) \in D$. □

Given a group G and a ring R, we use RG to denote the group ring of G over R.

Theorem 2.11. *A ring R is right permeable if and only if so is RG for any group G.*

Proof. Assume for the sake of contradiction that $ann_r(RG) \neq ann_r(R)G$, and let $\sum a_ig_i \in ann_r(RG)$, where some $a_i \notin ann_r(R)$. Then there exist $b \in R$ such that $ba_i \neq 0$, for some i. But $(be) \sum a_ig_i = \sum (ba_i)g_i = 0$, which implies that $ba_i = 0$ for all i, a contradiction. It is straightforward to see that $ann_l(RG) = ann_l(R)G$. Since R is right permeable, then $ann_r(RG) \subseteq ann_l(RG)$ and so RG is right permeable. For the converse, let $a \in ann_r(R)$ and $b \in R$. Then $(ae)(be) = abe = bae = 0$, hence R is right permeable. □

3 Permeable Rings of Matrices

More examples can be obtained from different rings and subrings of matrices. Let $M_n(R)$ be the complete matrix ring over a ring R. Let us also denote by $D_n(R)$ the set of diagonal matrices, $U_n(R)(L_n(R))$ the set of all upper (lower) triangular matrices, and $SU_n(R)$ $(SL_n(R))$ the set that contains all strictly upper (lower) triangular matrices. Note that $SU_n(R)$ and $SL_n(R)$ are not permeable rings for all non-zero rings R and $n \geq 3$. The following theorem shows positive responses from $M_n(R)$, $U_n(R)$, $L_n(R)$ and $D_n(R)$.

Theorem 3.1. *For a ring R, the following are equivalent:*

1. *R is right permeable.*
2. *$M_n(R)$ is right permeable.*
3. *$D_n(R)$ is right permeable.*
4. *$U_n(R)$ is right permeable.*
5. *$L_n(R)$ is right permeable.*

Proof. (1) $\iff$ (2) We don't know whether the following formula is known or not, we prefer to prove it here.

$$ann_r(M_n(R)) = \begin{bmatrix} ann_r(R) & \cdots & ann_r(R) \\ \vdots & & \vdots \\ ann_r(R) & \cdots & ann_r(R) \end{bmatrix} = M_n(ann_r(R))$$

Suppose on the contrary that $A \in ann_r(M_n(R))$, that is $BA = 0\ \forall B \in M_n(R)$, and $a_{ij} \notin ann_r(R)$ for some entries a_{ij} of A. Thus there exists $b_{ji} \in R$ such that $b_{ji}a_{ij} \neq 0$. Consider the matrix, say Q, with entries are all zero except the entry in the jth row and ith column and let it be b_{ji}. Then QA has the entry $b_{ji}a_{ij}$ in the jth row and jth column. But $b_{ji}a_{ij} \neq 0$. This contradicts that $QA = 0$. Analogously, we can prove that $ann_l(M_n(R)) = M_n(ann_l(R))$. Since R is right permeable, then $ann_r(M_n(R)) \subseteq ann_l(M_n(R))$. Therefore, $M_n(R)$ is right permeable.

Conversely, let $M_n(R)$ be right permeable where $a \in ann_r(R)$ and $r \in R$. Then

$$A = \begin{bmatrix} a & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \in ann_r(M_n(R)), \quad B = \begin{bmatrix} r & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \in M_n(R)$$

Hence $AB = BA = 0\ \forall B \in M_n(R) \Rightarrow ar = 0\ \forall r \in R$. Thus R is right permeable. In the same manner we can prove (1) $\iff$ (3), (1) $\iff$ (4) and (1) $\iff$ (5). ☐

Example 3.2.

Consequences of above results are appeared in following examples.

1. For any ring R, clearly, $SL_3(R)$ is symmetric and semi-commutative. But if we take

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & x & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

in $SL_3(\mathcal{A}_4)$, then $SL_3(\mathcal{A}_4)A = 0$, while $AB \neq 0$. Hence $SL_3(\mathcal{A}_4)$ is not right permeable or permeable. On the other hand, by Theorem 3.1, if R is right (or left) permeable, then so is $M_n(R)$. For instance, $\mathcal{A}_6$ is a proper permeable ring, so is $M_n\mathcal{A}_6)$. Again, clearly, $M_n(\mathcal{A}_6)$

is neither symmetric nor semi-commutative. Hence we have verified the parts of the Chart that

$$Symmetric \nLeftrightarrow Permeable$$
$$Semi-commutative \nLeftrightarrow Permeable$$

2. Another function of this example is to show that permeable rings need not be reversible and hence not fully symmetric, since $BA = 0$ but $AB \neq 0$. It is also clear that $SU_3(\mathcal{A}_4)$ and $SL_3(\mathcal{A}_4)$ are symmetric rings which are not fully symmetric.
3. Next, we notice that $SL_3(\mathcal{A}_4)$ is an ideal and also a subring of $L_3(\mathcal{A}_4)$, thus a subring or an ideal of a permeable ring may not be permeable.

Now we give example of nontrivial proper right permeable rings and another example for nontrivial proper left permeable rings. For a ring R, where $n \geq 2$ and $1 \leq k \leq r \leq n$. Consider

$$ZR_n^{k,r}(R) = \left\{ \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ a_{k1} & \cdots & a_{kn} \\ \vdots & & \vdots \\ a_{r1} & \cdots & a_{rn} \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix} : a_{ij} \in R \right\}.$$

Also, consider

$$ZC_n^{k,r}(R) = \left\{ \begin{bmatrix} 0 & \cdots & 0 & a_{1k} & \cdots & a_{1r} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & a_{nk} & \cdots & a_{nr} & 0 & \cdots & 0 \end{bmatrix} : a_{i,j} \in R \right\}.$$

Where $ZR_n^{k,r}(R)$ and $ZC_n^{k,r}(R)$ are subrings of $M_n(R)$.

Theorem 3.3. *Let R be a right permeable ring. Then $ZR_n^{k,r}(R)$ is a right permeable ring.*

Proof. Clearly $ann_r(ZR_n^{k,r}(R)) = ZR_n^{k,r} ann_r(R)$, and $ann_l(ZR_n^{k,r}(R)) =$

$$\left\{ \begin{bmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ a_{k1} & \cdots & a_{k,k-1} & b_{kk} & \cdots & b_{kr} & a_{k,r+1} & \cdots & a_{kn} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a_{r1} & \cdots & a_{r,k-1} & b_{rk} & \cdots & b_{rr} & a_{r,r+1} & \cdots & a_{rn} \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix} : \begin{array}{c} a_{ij} \in R \\ b_{st} \in ann_l(R) \end{array} \right\}.$$

Since $ann_r(R) \subseteq ann_l(R)$. So $ann_r(ZR_n^{k,r}(R)) \subseteq ann_l(ZR_n^{k,r}(R))$ and hence $ZR_n^{k,r}(R)$ is a right permeable ring. ☐

Corollary 3.4. *Let R be a non-zero right permeable ring. If $k \neq 1$ or $r \neq n$, then $ZR_n^{k,r}(R)$ is a proper right permeable ring. Moreover, if R is non-trivially right permeable, then so is $ZR_n^{k,r}(R)$.*

Proof. Since R is a non-zero right permeable ring, then there exists $a \in R$ such that $a \notin ann_r(R)$. If $k \neq 1$ or $r \neq n$, then take the matrix A with entries are all zero except $a_{k1} = a$ or $a_{rn} = a$, respectively. Thus $A \in ann_l(ZR_n^{k,r}(R))$ but $A \notin ann_r(ZR_n^{k,r}(R))$ and so $ann_r(ZR_n^{k,r}(R)) \subsetneq ann_l(ZR_n^{k,r}(R))$. Therefore $ZR_n^{k,r}(R)$ is a proper right permeable ring. ☐

Theorem 3.5. *Let R be a left permeable ring. Then $ZC_n^{k,r}(R)$ is a left permeable ring.*

Proof. It is clear that $ann_l(ZC_n^{k,r}(R)) = ZC_n^{k,r}(ann_l(R))$, and $ann_r(ZC_n^{k,r}(R)) =$

$$\left\{ \begin{bmatrix} 0 & \cdots & 0 & a_{1k} & \cdots & a_{1r} & 0 & \cdots & 0 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ 0 & \cdots & 0 & a_{k-1,k} & \cdots & a_{k-1,r} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & b_{kk} & \cdots & b_{kr} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & b_{rk} & \cdots & b_{rr} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & a_{r+1,k} & \cdots & a_{r+1,r} & 0 & \cdots & 0 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ 0 & \cdots & 0 & a_{nk} & \cdots & a_{nr} & 0 & \cdots & 0 \end{bmatrix} : \begin{array}{c} a_{ij} \in R \\ b_{st} \in ann_r(R) \end{array} \right\}.$$

Since $ann_l(R) \subseteq ann_r(R)$. Therefore $ann_l(ZC_n^{k,r}(R)) \subseteq ann_r(ZC_n^{k,r}(R))$ and hence $ZC_n^{k,r}(R)$ is a left permeable ring. ☐

Corollary 3.6. *Let R be a non-zero left permeable ring. If $k \neq 1$ or $r \neq n$, then $ZC_n^{k,r}(R)$ is a proper left permeable ring. Moreover, if R is non-trivially left permeable, then $ZC_n^{k,r}(R)$ is non-trivially left permeable.*

Proof. Since R is a non-zero left permeable ring, then there exists $a \in R$ such that $a \notin ann_l(R)$. If $k \neq 1$ or $r \neq n$, then take the matrix A with entries are all zero except $a_{1k} = a$ or $a_{nr} = a$, respectively. Thus $A \in ann_r(ZC_n^{k,r}(R))$ but $A \notin ann_l(ZC_n^{k,r}(R))$ and so $ann_l(ZC_n^{k,r}(R)) \subsetneq ann_r(ZC_n^{k,r}(R))$. Therefore $ZC_n^{k,r}(R)$ is a proper left permeable ring. ☐

Remark 3.1. If R is a proper right permeable ring with $k \neq 1$ or $r \neq n$, then $ZC_n^{k,r}(R)$ is not permeable. For instance, consider $R = ZC_3^{2,3}(\mathcal{V}_4^{op})$. Take

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & c & c \\ 0 & c & c \end{bmatrix}, \quad A = \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then $BR = 0$ while $RB \neq 0$. On the other hand $AR \neq 0$ but $RA = 0$. Hence R is not permeable. This is also true for $ZR_n^{k,r}(R)$, if R is left permeable.

Now we introduce a type of matrices to show that factor ring of a permeable ring need not be permeable. Let R be a ring and $n \geq 3$, where $1 \leq k \leq r \leq n$. Then we define a subring $ZA_n^{k,r}(R)$ of $M_n(R)$

$$ZA_n^{k,r}(R) = \left\{ \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ a_{k-1,1} & 0 & \cdots & 0 \\ a_{k1} & a_{k2} & \cdots & a_{kn} \\ \vdots & \vdots & & \vdots \\ a_{r,1} & a_{r,2} & \cdots & a_{rn} \\ a_{r+1,1} & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ a_{n1} & 0 & \cdots & 0 \end{bmatrix} : a_{ij} \in R \right\}$$

Theorem 3.7. *Let R be a ring. Then R is right permeable if and only if $ZA_n^{k,r}(R)$ is right permeable.*

Proof. It is straightforward to verify that $ann_r(ZA_n^{k,r}(R)) = ZA_n^{k,r}(ann_r(R))$ and $ann_l(ZA_n^{k,r}(R)) = ZA_n^{k,r}(ann_l(R))$. Since R is right permeable, then $ann_r(R) \subseteq ann_l(R)$ and so $ann_r(ZA_n^{k,r}(R)) \subseteq ann_l(ZA_n^{k,r}(R))$. The converse is clear. □

Example 3.8. *Consider $ZA_3^{2,2}(\mathcal{A}_4)$. Then*

$$I = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & 0 & 0 \end{bmatrix} : a_{21}, a_{31} \in \mathcal{A}_4 \right\}$$

is an ideal of $ZA_3^{2,3}(\mathcal{A}_4)$. Which implies that

$$ZA_3^{2,3}(\mathcal{A}_4)/I \cong R = \left\{ \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & 0 & 0 \end{bmatrix} : a_{11}, a_{22}, a_{23} \in \mathcal{A}_4 \right\}.$$

But R is not permeable, since if we take the matrix A with entries are all zero except the entry $a_{23} = x$. Then $AR = 0 \neq RA$.

For a ring R and a completely reflexive ideal I (i.e. if $xy \in I$, then $yx \in I$, for $x, y \in R$, see [10] for more on this topic). Let $a + I \in ann_r(R/I)$. Then $ra \in I$, $\forall r \in R$, which implies that $ar \in I$, $\forall r \in R$. Thus $ann_r(R/I) \subseteq ann_l(R/I)$. Similarly, $ann_l(R/I) \subseteq ann_r(R/I)$. So we proved the following:

Theorem 3.9. *If R is a ring and I is a completely reflexive ideal, then the factor ring R/I is permeable.*

4 Polynomial Related Extensions

Let R be a ring and x a commutative indeterminate over R. As usual we denote by $R[[x]]$ and $R[x; x^{-1}]$ the power series and Laurent polynomial rings, respectively. Then we have:

Theorem 4.1. *For any ring R the following are equivalent:*

1. *R is right permeable.*
2. *$R[x]$ is right permeable.*
3. *$R[[x]]$ is right permeable.*
4. *$R[x; x^{-1}]$ is right permeable.*

Proof. (1) $\iff$ (2) First we show that $ann_r(R[x]) = ann_r(R)[x]$. Suppose for the sake of contradiction that $ann_r(R[x]) \neq ann_r(R)[x]$ and let $p(x) = a_0 + a_1x + \cdots + a_nx^n \in ann_r(R[x])$ such that $a_i \notin ann_r(R)$ for some $0 \leq i \leq n$. Then there exists $b_i \in R$ such that $b_ia_i \neq 0$ and hence $q(x)p(x) \neq 0$, where $q(x) = b_ix^i$, a contradiction. Let R be a right permeable ring. Let $p(x) = a_0 + a_1x + \cdots + a_nx^n \in ann_r(R)[x]$ and $q(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x]$. Then the coefficients of $p(x)q(x)$ are $\sum_{i=0}^{n+m} a_ib_{n+m-i} = 0$, $a_i \in ann_r(R)$ and so $R[x]$ is right permeable. Conversely, let $R[x]$ be a right permeable ring. Take $a \in ann_r(R)$, then $p(x) = a \in ann_r(R[x])$ and $ab = ba = 0$ for all $b \in R[x]$. Thus $ab = ba = 0$ for all $b \in R$. In the same manner we can prove that (1) $\iff$ (3) and (1) $\iff$ (4). □

Let R be a ring and α an epimorphism on R. If $a \in ann_r(R)$, then $R\alpha(a) = \alpha(R)\alpha(a) = \alpha(Ra) = 0$. Thus $\alpha(a) \in ann_r(R)$. Also, if δ is an α-derivation, then $0 = \delta(Ra) = \delta(R)\alpha(a) + R\delta(a) = R\delta(a)$, and so $\delta(a) \in ann_r(R)$.

Now let x be a not necessarily commutative indeterminate and let $R[x; \alpha, \delta]$ be the *Ore extension ring* in which we define the commutation formula by the rule: $\forall a \in R,\ xa = \alpha(a)x + \delta(a)$. Then we have:

Theorem 4.2. *A ring R is right permeable if and only if $R[x; \alpha, \delta]$ is right permeable.*

Proof. We will prove that $ann_r(R[x; \alpha, \delta]) = ann_r(R)[x; \alpha, \delta]$. Clearly

$$ann_r(R)[x; \alpha, \delta] \subseteq ann_r(R[x; \alpha, \delta]),$$

since if $r(x) = r_0 + r_1x + \cdots + r_nx^n \in ann_r(R)[x; \alpha, \delta]$, then $s(x)r(x) = 0\ \forall s(x) \in R[x; \alpha, \delta]$, where $\delta^k(\alpha^l(r_i))$ and $\alpha^u(\delta^v(r_i)) \in ann_r(R)\ \forall i, k, l, u, v \geq 0$. Suppose for the sake of contradiction that $ann_r(R[x; \alpha, \delta]) \not\subset ann_r(R)[x; \alpha, \delta]$ and let $p(x) = a_0 + a_1x + \cdots + a_nx^n \in ann_r(R[x; \alpha, \delta])$ with some coefficients a_i not in $ann_r(R)$, $1 \leq i \leq n$. Then there exists $b \in R$ such that $ba_i \neq 0$, for some i. Take $q(x) = b$ and so $q(x)p(x) = 0$ implies that $ba_0 + ba_1x + \cdots + ba_nx^n = 0$. So $ba_j = 0\ \forall j = 0, 1, \ldots n$, a contradiction. Now, let $a(x) \in ann_r(R[x; \alpha, \delta])$ and $b(x) \in R[x; \alpha, \delta]$. Since R is right permeable, then $a(x)b(x) = 0$. Which means that $a(x) \in ann_l(R[x; \alpha, \delta])$ and so $R[x; \alpha, \delta]$ is right permeable. The converse is clear. □

For a ring R, and a commutative indeterminate x, $x^kR[x]$ denotes the ideal of $R[x]$ consisting the zero polynomial and all polynomials of the form $\sum_{j=0}^{n} a_jx^{k+j}$, where $a_j \in R$, and $k \geq 1$. We start by showing that the Barnett matrix ring, denoted by $M_k(R; x^k)$ and defined by

$$M_k(R; x^k) = \left\{ \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{k-1} \\ 0 & a_0 & a_1 & \cdots & a_{k-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_0 \end{bmatrix} : a_i \in R,\ 0 \leq i \leq k-1 \right\},$$

is permeable. Note that $M_k(R; x^k) \cong R[x]/x^kR[x]$.

Theorem 4.3. *A ring R is right permeable if and only if $M_k(R; x^k)$ is right permeable.*

Proof. The proof of $ann_r(M_k(R; x^k)) = M_k(ann_r(R); x^k)$ and if R is right permeable implies that $M_k(R; x^k)$ is right permeable can be done in the same way as in Theorem 3.1. For the converse, let $M_k(R; x^k)$ be right permeable, $a \in ann_r(R)$ and $r \in R$. Consider the matrices

$$A = \begin{bmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & a & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 & \cdots & r \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Since $AB = BA = 0$, then $ar = 0\ \forall r \in R$. Thus R is right permeable. □

Hence we also have deduced that:

Corollary 4.4. *A ring R is right permeable if and only if the factor polynomial ring $R[x]/x^kR[x]$, $\forall k \geq 1$, is right permeable.*

Next we define an extension of the Barnett matrix ring in the following form:

Definition 4.5. For any right permeable ring R, we define

$$\overline{M_k}(R;x^k) = \left\{ \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{k-1} \\ b_{1,1} & a_0 & a_1 & \cdots & a_{k-2} \\ b_{2,1} & b_{2,2} & a_0 & & \vdots \\ \vdots & \vdots & \vdots & & a_1 \\ b_{k-1,1} & b_{k-1,2} & \ldots & b_{k-1,k-1} & a_0 \end{bmatrix} : \begin{array}{c} a_i \in R \\ b_{l,j} \in ann_r(R) \\ i = 0, ..., k-1 \\ l = 1, ..., k-1 \\ j = 1, ..., k-1 \end{array} \right\}$$

Clearly, $\overline{M_k}(R;x^k)$ is a subring of $M_k(R)$.

Note that the indeterminate x has no role in the structure of $\overline{M_k}(R;x^k)$, it is there just to remind its link with the factor polynomial ring $R[x]/x^kR[x]$, like in the case of the Barnett matrix ring itself.

Remark 4.1. Let $\overline{A}, \overline{B} \in \overline{M_k}(R;x^k)$ and $A, B \in M_k(R;x^k)$, where R is a right permeable ring. If the upper triangular part of $\overline{A}$ is equal to that of A, and same for $\overline{B}$ and B, then $\overline{A}\,\overline{B} = AB$.

Obviously, in general,

$$\overline{M_k}(R;x^k) \ncong R[x]/x^kR[x].$$

In the following we develop a link between these notions in the presence of permeability property of R.

Theorem 4.6. *If R is a right permeable ring, then $R[x]/x^kR[x]$ is a homomorphic image of $\overline{M_k}(R;x^k)$.*

Proof. Define a map

$$\varphi : \overline{M_k}(R,x^k) \longrightarrow R[x]/x^kR[x]$$

by

$$\varphi\left(\begin{bmatrix} a_0 & a_1 & \cdots & a_{k-1} \\ b_{1,1} & a_0 & \cdots & a_{k-2} \\ \vdots & \vdots & & a_1 \\ b_{k-1,1} & b_{k-1,2} & \ldots & a_0 \end{bmatrix}\right) = a_0 + a_1x + \cdots + a_{k-1}x^{k-1}.$$

Clearly, φ is well-defined and onto. Let $A, B \in \overline{M_k}(R,x^k)$ such that

$$A = \begin{bmatrix} a_0 & a_1 & \cdots & a_{k-1} \\ b_{1,1} & a_0 & \cdots & a_{k-2} \\ \vdots & \vdots & & \vdots \\ b_{k-1,1} & b_{k-1,2} & \ldots & a_0 \end{bmatrix},\ B = \begin{bmatrix} c_0 & c_1 & \cdots & c_{k-1} \\ d_{1,1} & c_0 & \cdots & c_{k-2} \\ \vdots & \vdots & & \vdots \\ d_{k-1,1} & d_{k-1,2} & \cdots & c_0 \end{bmatrix}.$$

Then $\varphi(A+B) = \varphi(A) + \varphi(B)$. Also

$$\varphi(AB) = \begin{bmatrix} e_0 & e_1 & \cdots & e_{k-1} \\ 0 & e_0 & \cdots & e_{k-2} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & e_0 \end{bmatrix}, \quad e_n = \sum_{i=0}^{n} a_ic_{n-i},\ 0 \le n \le k-1$$

Thus

$$\begin{aligned} \varphi(AB) &= e_0 + \cdots + e_{k-1}x^{k-1} = (a_0 + \cdots + a_{k-1}x^{k-1})(c_0 + \cdots + c_{k-1}x^{k-1}) \\ &= \varphi(A)\varphi(B). \end{aligned}$$

This proves that φ is an epimorphism. □

For a right (left) permeable ring R, the subrings $\overline{D_n}(R)$, $\overline{U_n}(R)$, $\overline{L_n}(R)$, $\overline{SU_n}(R)$, $\overline{SL_n}(R)$, $\overline{ZR_n^{k,r}}(R)$, $\overline{ZC_n^{k,r}}(R)$ and $\overline{ZA_n^{k,r}}(R)$ of $M_n(R)$ in the sense of Theorem 4.6 are extensions to the homomorphic images $D_n(R)$, $U_n(R)$, $L_n(R)$, $SU_n(R)$, $SL_n(R)$, $ZR_n^{k,r}(R)$, $ZC_n^{k,r}(R)$ and $ZA_n^{k,r}(R)$, respectively. For example,

$$\overline{D_n}(R) = \left\{ \begin{bmatrix} a_1 & b_{12} & \dots & b_{1n} \\ b_{21} & a_2 & \dots & b_{2n} \\ b_{31} & b_{32} & \cdots & b_{3n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \dots & a_n \end{bmatrix} : a_i \in R, b_{rj} \in ann_r(R) \right\}.$$

Definition 4.7. Let R be a right (left) permeable ring. A *right (left) permeable power series*, in short *rp- series (lp-series)* $\overline{f(x)}$ with coefficients in R is an infinite formal sum

$$\sum_{i=0}^{\infty} a_i x^i = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \dots$$

where $a_i \in R$ and $a_i \in ann_r(R)$ for all but a finite number of i. The a_i are coefficients of $\overline{f(x)}$. If for some $i \geq 0$ it is true that $a_i \notin ann_r(R)$, the largest such value of i is the degree of $\overline{f(x)}$. If all $a_i \in ann_r(R)$, then the degree of $\overline{f(x)}$ is undefined.

Theorem 4.8. *The set $\overline{R[x]}$ of all rp-series in a commutative indeterminate x with coefficients from a right permeable ring R is a subring of the power series $R[[x]]$. If R is trivially right permeable, then $\overline{R[x]} = R[x]$. If R is non-trivially right permeable, then $R[x]$ is an ideal of $\overline{R[x]}$.*

Proof. Let $\overline{f(x)}, \overline{g(x)} \in \overline{R[x]}$. Then, clearly $\overline{f(x)} + \overline{g(x)} \in \overline{R[x]}$ and $\overline{f(x)} \cdot \overline{g(x)} \in \overline{R[x]}$, since all the coefficients of $\overline{f(x)}$ and $\overline{g(x)}$ are 0 except for a finite number of them. If R is trivially right permeable, then zero is the only element in $ann_r(R)$ and so $\overline{R[x]} = R[x]$. Finally, if R non-trivially right permeable and $\overline{f(x)} \in \overline{R[x]}, g(x) \in R[x]$, then also the coefficients of $\overline{f(x)}g(x)$ and $g(x)\overline{f(x)}$ are all 0 except for a finite number of them. Thus $\overline{f(x)}g(x), g(x)\overline{f(x)} \in R[x]$. □

Let R be a right (left) permeable ring and $\overline{f(x)} \in \overline{R[x]} \backslash R[x]$. If $\overline{f(x)}$ has undefined degree, then we say that $\overline{f(x)}$ is a pseudo-zero polynomial. While $\overline{f(x)} = \sum_{i=0}^{\infty} a_i x^i$ with degree k may be termed as pseudo-$f(x)$ polynomial if $f(x) = \sum_{i=0}^{k} b_i x^i \in R[x]$ with degree k and $a_j = b_j$ for all $0 \leq j \leq k$.

Example 4.9. *A counter case to Theorem 4.6.*

Similar to Remark 4.1, if $\overline{f(x)}$ and $\overline{g(x)}$ are pseudo-$f(x)$ and pseudo-$g(x)$ in $\overline{R[x]}$, respectively, where R is a right permeable ring, then $\overline{f(x)} \cdot \overline{g(x)} = f(x)g(x)$. But opposite to the Theorem 4.6, if we take the onto map $\varphi : \overline{R[x]} \longrightarrow R[x]$ given by

$$\varphi(\overline{f(x)}) = \begin{cases} \overline{f(x)} & \text{if } \overline{f(x)} \in R[x] \\ p(x) & \text{if } \overline{f(x)} \text{ is pseudo} - \text{zero} \\ f(x) & \text{if } \overline{f(x)} \text{ is pseudo} - f(x) \end{cases}$$

where $p(x) \in ann_r(R[x])$ is not a homomorphism (in general). Suppose on the contrary that φ is a homomorphism and R is a non-trivially non-zero right permeable ring. Consider $\overline{f(x)} = \sum_{i=0}^{\infty} bx^i$, where $0 \neq b \in ann_r(R)$, and so $\overline{f(x)}$ is a pseudo-zero. Thus $\varphi(\overline{f(x)}) = \sum_{i=0}^{k} b_i x^i, b_i \in ann_r(R)$. Let $f(x) = \sum_{i=0}^{k+1} a_i x^i$ be a polynomial of degree $k+1$ which implies that $\overline{f(x)} + f(x)$ is a pseudo-$(\overline{f(x)} + f(x))$. Then

$$\begin{aligned} \varphi(\overline{f(x)} + f(x)) &= \sum_{i=0}^{k+1} (b + a_i) x^i = \varphi(\overline{f(x)}) + \varphi(f(x)) \\ &= \sum_{i=0}^{k} (b_i + a) x^i + a_{k+1} x^{k+1}. \end{aligned}$$

Which implies that $b + a_{k+1} = a_{k+1}$, and so $b = 0$, a contradiction.

5 Conclusion

As we have seen, the permeability property respects many ring extensions and can lead to many useful extensions. Especially in the polynomial ring where two permeable power series act like two polynomials. Also we extend the Barnett matrix ring and so the factor ring $R[x]/x^k R[x]$.

Competing Interests

Authors have declared that no competing interests exist.

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