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Evaluation of Certain Convolution Sums Involving Divisor Functions and Infinite Product Sums

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Abstract

Originating from the convolution sum

$$\sum_{m<\frac{n}{8}} \sigma_3(m)\sigma_3(n-8m)$$

for $n \in \mathbb{N}$ by Kim's result, we try to induce the convolution sum formulae as

$$\sum_{m<\frac{n}{8}} \sigma_1(m)\sigma_5(n-8m) \quad \text{and} \quad \sum_{m<\frac{n}{8}} \sigma_5(m)\sigma_1(n-8m)$$

therefore we obtain the desired results. Moreover we construct some various convolution sums and obtain their formulae.

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1 Introduction

The study of arithmetical identities is classical in number theory and such investigations have been carried out by several mathematicians including the legend Srinivasa Ramanujan.

For $n \in \mathbb{N}$, $s \in \mathbb{N} \cup \{0\}$, $q \in \mathbb{C}$ with $|q| < 1$, we define the divisor function and the infinite product sums :

$$\begin{aligned} \sigma_s(n) = \sum_{d|n} d^s, \quad \Delta(q) &:= \sum_{n=1}^{\infty} \tau(n) q^n = q \prod_{n=1}^{\infty} (1-q^n)^{24}, \\ B(q) := \sum_{n=1}^{\infty} b(n) q^n &= (\Delta(q)\Delta(q^2))^{\frac{1}{3}} = q \prod_{n=1}^{\infty} (1-q^n)^8 (1-q^{2n})^8, \\ G(q) := \sum_{n=1}^{\infty} g(n) q^n = 2^4 \left(\frac{\Delta(q^2)^{11}}{\Delta(q^4)^3 \Delta(q)^4} \right)^{\frac{1}{6}} &= 2^4 q \prod_{n=1}^{\infty} \frac{(1+q^n)^{32}(1-q^n)^{16}}{(1+q^{2n})^{12}}, \\ H(q) := \sum_{n=1}^{\infty} h(n) q^n = 2^{12} \left(\frac{\Delta(q^4)^5}{\Delta(q^2)} \right)^{\frac{1}{6}} &= 2^{12} q^3 \prod_{n=1}^{\infty} (1+q^{2n})^4 (1-q^{4n})^{16}. \end{aligned} \tag{1.1}$$

In general, it is satisfied that

$$b(n) = -8b(\frac{n}{2}) \qquad \text{and} \qquad h(n) = 0 \tag{1.2}$$

for even n (see [1], [2], ([3] Remark 4.3)). As an extension of (1.2) we obtain the following lemma.

Lemma 1.1. *Let $n \in \mathbb{N}$ be an even positive integer. Then we have*

$$g(n) = 256b(\frac{n}{2}) + 8192b(\frac{n}{4}).$$

For $q \in \mathbb{C}$ satisfying $|q| < 1$, the Eisenstein series $L(q)$, $M(q)$, and $N(q)$ are

$$L(q) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n, \tag{1.3}$$

$$M(q) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n, \tag{1.4}$$

$$N(q) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n, \tag{1.5}$$

see ([4], p. 318). It was shown that

$$\Delta(q) = \frac{1}{1728} \left(M(q)^3 - N(q)^2 \right) \tag{1.6}$$

by Ramanujan. And he gave in his notebook the following formulae, which are proved in ([5], p. 126-129):

$$L(q) = (1-5x)w^2 + 12x(1-x)w\frac{dw}{dx}, \tag{1.7}$$

$$M(q) = (1 + 14x + x^2) w^4, \tag{1.8}$$

$$N(q) = (1+x)(1-34x+x^2) w^6, \tag{1.9}$$

$$L(q^2) = (1-2x)w^2 + 6x(1-x)w\frac{dw}{dx}, \tag{1.10}$$

$$M(q^2) = (1-x+x^2)w^4, \tag{1.11}$$

$$N(q^2) = (1+x)(1-\frac{1}{2}x)(1-2x)w^6, \tag{1.12}$$

$$L(q^4) = (1-\frac{5}{4}x)w^2 + 3x(1-x)w\frac{dw}{dx}, \tag{1.13}$$

$$M(q^4) = (1-x+\frac{1}{16}x^2)w^4, \tag{1.14}$$

$$N(q^4) = (1-\frac{1}{2}x)(1-x-\frac{1}{32}x^2)w^6, \tag{1.15}$$

where for $0 < x < 1$, w is defined by

$$w = {}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x) = \sum_{n=0}^{\infty} \frac{1}{2^{4n}} \binom{2n}{n}^2 x^n$$

with the Gaussian hypergeometric function ${}_2F_1(a, b; c; x)$. From (1.6), (1.8), and (1.9), we obtain

$$\Delta(q) = \frac{x(1-x)^4 w^{12}}{2^4}. \tag{1.16}$$

Applying the principle of duplication (see ([5], p. 125))

$$q \to q^2, \qquad x \to \left(\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right)^2, \qquad w \to \left(\frac{1+\sqrt{1-x}}{2}\right) w$$

to (1.16), we induce that

$$\Delta(q^2) = \frac{x^2(1-x)^2 w^{12}}{2^8}. \tag{1.17}$$

Again applying the principle of duplication to (1.17) and (1.14), respectively we have

$$\Delta(q^4) = \frac{x^4(1-x)w^{12}}{2^{16}}$$

and

$$\begin{aligned} M(q^8) &= (\frac{17}{32} - \frac{17}{32}x + \frac{1}{256}x^2 + \frac{15}{32}\sqrt{1-x} - \frac{15}{64}x\sqrt{1-x})w^4 \\ &= -\frac{1}{32}M(q^2) + \frac{9}{16}M(q^4) + \frac{15}{32}\sqrt{1-x}w^4 - \frac{15}{64}x\sqrt{1-x}w^4. \end{aligned} \tag{1.18}$$

From the above information Kim showed that

$$G(q) = x\sqrt{1-x}w^8 \quad \text{and} \quad H(q) = x^3\sqrt{1-x}w^8 \tag{1.19}$$

in ([1], (2.1), (2.2)). In fact, since the convolution sum formula as the form

$$\sum_{m<\frac{n}{8}} \sigma_3(m)\sigma_3(n-8m)$$
$$= \frac{1}{8355840}\Big\{16\sigma_7(n) + 240\sigma_7(\frac{n}{2}) + 3840\sigma_7(\frac{n}{4}) + 65536\sigma_7(\frac{n}{8})$$
$$- 34816\sigma_3(n) - 34816\sigma_3(\frac{n}{8}) + 18480b(n) + 197760b(\frac{n}{2})$$
$$-3624960b(\frac{n}{4}) + 1020g(n) - 255h(n)\Big\}$$

(see ([1], Theorem 3.2)) has already obtained with all $n \in \mathbb{N}$ therefore in this paper, we are willing to evaluate the similar convolution sum formulae and obtain :

Theorem 1.2. *Let* $n \in \mathbb{N}$. *Then we have*

(a)

$$\sum_{m<\frac{n}{8}} \sigma_1(m)\sigma_5(n-8m)$$
$$= \frac{1}{2193408}\Big\{1344\sigma_7(n) + 4032\sigma_7(\frac{n}{2}) + 16128\sigma_7(\frac{n}{4}) + 65536\sigma_7(\frac{n}{8})$$
$$- 22848\,(n-4)\,\sigma_5(n) + 4352\sigma_1(\frac{n}{8}) - 1071h(n) - 2142g(n) - 35616b(n)$$
$$-1421952b(\frac{n}{2}) - 18665472b(\frac{n}{4})\Big\}.$$

In particular, we can simplify

$$\sum_{m<\frac{n}{8}} \sigma_1(m)\sigma_5(n-8m)$$
$$= \begin{cases} \frac{1}{34272}\Big\{21\sigma_7(n) + 63\sigma_7(\frac{n}{2}) + 252\sigma_7(\frac{n}{4}) + 1024\sigma_7(\frac{n}{8}) \\ \quad -357\,(n-4)\,\sigma_5(n) + 68\sigma_1(\frac{n}{8}) - 26334b(\frac{n}{2}) \\ \quad -565824b(\frac{n}{4})\Big\}, & \textit{for even } n, \\ \frac{1}{104448}\{64\sigma_7(n) - 1088\,(n-4)\,\sigma_5(n) - 51h(n) - 102g(n) \\ \quad -1696b(n)\}, & \textit{for odd } n, \end{cases}$$

(b)

$$\sum_{m<\frac{n}{8}} \sigma_5(m)\sigma_1(n-8m)$$
$$= \frac{1}{280756224}\Big\{32\sigma_7(n) + 2016\sigma_7(\frac{n}{2}) + 129024\sigma_7(\frac{n}{4}) + 11010048\sigma_7(\frac{n}{8})$$
$$- 11698176\,(2n-1)\,\sigma_5(\frac{n}{8}) + 557056\sigma_1(n) + 7497h(n) - 17136g(n)$$
$$-282912b(n) + 446208b(\frac{n}{2}) + 122142720b(\frac{n}{4})\Big\}.$$

In particular, we can simplify

$$\sum_{m<\frac{n}{8}} \sigma_5(m)\sigma_1(n-8m)$$
$$= \begin{cases} \frac{1}{8773632}\Big\{\sigma_7(n) + 63\sigma_7(\frac{n}{2}) + 4032\sigma_7(\frac{n}{4}) + 344064\sigma_7(\frac{n}{8}) \\ \quad -365568\,(2n-1)\,\sigma_5(\frac{n}{8}) + 17408\sigma_1(n) - 52416b(\frac{n}{2}) \\ \quad -569856b(\frac{n}{4})\Big\}, & \textit{for even } n, \\ \frac{1}{280756224}\{32\sigma_7(n) + 557056\sigma_1(n) + 7497h(n) - 17136g(n) \\ \quad -282912b(n)\}, & \textit{for odd } n. \end{cases}$$

Furthermore, by defining

$$D(q) := \sum_{n=1}^{\infty} d(n)q^n = \big(\Delta(q)^2\Delta(q^2)\Delta(q^4)^2\big)^{\frac{1}{6}} = q^2\prod_{n=1}^{\infty}(1-q^n)^8(1-q^{2n})^4(1-q^{4n})^8,$$
$$F(q) := \sum_{n=1}^{\infty} f(n)q^n = \left(\frac{\Delta(q^4)^4}{\Delta(q)}\right)^{\frac{1}{3}} = q^5\prod_{n=1}^{\infty}(1-q^n)^{24}(1+q^n)^{32}(1+q^{2n})^{32}$$

and by Lemma 1.1 we can generalize some convolution sums :

Theorem 1.3. *Let* $n, k \in \mathbb{N}$ *with* $k \geq 2$*. Then we have*

(a)

$$\sum_{m=1}^{n-1} g(2^k m)\sigma_1(n-m) = 2(-2)^{3k}\{d(2n) - (3n-2)\,b(n)\},$$

(b)

$$\sum_{m<\frac{n}{2}} g(2^k m)\sigma_1(n-2m) = \begin{cases} -(-2)^{3k+1}\Big\{4d(n) - (3n-2)\,b(\frac{n}{2})\Big\}, & \textit{for even } n, \\ 6(-2)^{3k+1}d(n), & \textit{for odd } n, \end{cases}$$

(c)

$$\sum_{m<\frac{n+1}{2}} g(2^k(2m-1))\sigma_1(n-2m+1)$$
$$= \begin{cases} 6(-2)^{3k+4}d(n), & \textit{for even } n, \\ -(-2)^{3k+1}\{d(2n) - 48d(n) - (3n-2)\,b(n)\}, & \textit{for odd } n, \end{cases}$$

(d)

$$\sum_{m=1}^{n-1} g(2^k m)\sigma_3(n-m) = -\frac{1}{5}(-2)^{3k+1}\left\{\tau(n) + 256\tau(\frac{n}{2}) - b(n)\right\},$$

(e)

$$\sum_{m<\frac{n}{2}} g(2^k m)\sigma_3(n-2m)$$
$$= \begin{cases} -\frac{1}{5}(-2)^{3k+1}\left\{\tau(\frac{n}{2}) + 2176\tau(\frac{n}{4}) - b(\frac{n}{2})\right\}, & \textit{for even } n, \\ -\frac{3}{691}(-2)^{3k-3}\left\{\sigma_{11}(n) - \tau(n) - 45285376 f(n)\right\}, & \textit{for odd } n, \end{cases}$$

(f)

$$\sum_{m<\frac{n}{2}} g(2^k(2m-1))\sigma_3(n-2m+1)$$
$$= \begin{cases} -3(-2)^{3k+5}\left\{\tau(\frac{n}{2}) + 64\tau(\frac{n}{4})\right\}, & \textit{for even } n, \\ \frac{1}{3455}(-2)^{3k}\{15\sigma_{11}(n) + 1367\tau(n) - 679280640 f(n) & \\ \qquad -1382 b(n)\}, & \textit{for odd } n. \end{cases}$$

2 Preliminary Results

We can put

$$A(q) := \sum_{n=1}^{\infty} a(n)q^n = \Delta(q^2)^{\frac{1}{2}} = q\prod_{n=1}^{\infty}(1-q^{2n})^{12},$$
$$C(q) := \sum_{n=1}^{\infty} c(n)q^n = \left(\Delta(q)^4\Delta(q^2)\right)^{\frac{1}{6}} = q\prod_{n=1}^{\infty}(1-q^n)^{16}(1-q^{2n})^4 \tag{2.1}$$

then using (1.17) and (2.1) we have

$$A(q) = \frac{x(1-x)w^6}{16} \tag{2.2}$$

(refer to ([6], (4.1))). For all $n \in \mathbb{N}$ we have shown that

$$c(n) = d(2n) - 32d(n) \tag{2.3}$$

in ([7], Theorem 1.1). On the other hand, for only even n we can see that

$$a(n) = 0 \qquad \text{and} \qquad c(\frac{n}{2}) = d(n) - 32d(\frac{n}{2}) \tag{2.4}$$

(see ([7], (2.4))) and also we obtained Proposition 2.1 :

Proposition 2.1. *(See ([8], Theorem 1.1)) Let* $n\ (\in \mathbb{N})$ *be an even positive integer. Then we have*

(a)

$$d(n) = \frac{1}{16} d(2n),$$

(b)

$$f(n) = \frac{1}{45285376} \left\{ \sigma_{11}(n) - \sigma_{11}(\frac{n}{2}) - 2048\tau(\frac{n}{2}) - 4243456\tau(\frac{n}{4}) \right\},$$

(c)

$$\tau(n) = -24\tau(\frac{n}{2}) - 2048\tau(\frac{n}{4}).$$

Now we need the following identities which can be found in Lahiri ([9], p. 149)

$$L^2(q) = 1 - 288 \sum_{n=1}^{\infty} n\sigma_1(n)q^n + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n, \tag{2.5}$$

$$M^2(q) = 1 + 480 \sum_{n=1}^{\infty} \sigma_7(n)q^n, \tag{2.6}$$

$$\begin{aligned} L(q)M(q) &= 1 + 720 \sum_{n=1}^{\infty} n\sigma_3(n)q^n - 504 \sum_{n=1}^{\infty} \sigma_5(n)q^n, \\ L(q)M^2(q) &= 1 + 720 \sum_{n=1}^{\infty} n\sigma_7(n)q^n - 264 \sum_{n=1}^{\infty} \sigma_9(n)q^n, \\ L(q)N(q) &= 1 - 1008 \sum_{n=1}^{\infty} n\sigma_5(n)q^n + 480 \sum_{n=1}^{\infty} \sigma_7(n)q^n, \\ M(q)N(q) &= 1 - 264 \sum_{n=1}^{\infty} \sigma_9(n)q^n. \end{aligned} \tag{2.7}$$

For $e, f, m, n \in \mathbb{N}$ we set

$$\begin{aligned} I_{e,f}(n) &:= \sum_{m=1}^{n-1} \sigma_e(m)\sigma_f(n-m), \\ T_{e,f}(n) &:= \sum_{m<\frac{n}{2}} \sigma_e(m)\sigma_f(n-2m), \\ T_{m,e,f}(n) &:= \sum_{m<\frac{n}{2}} m\sigma_e(m)\sigma_f(n-2m), \\ U_{e,f}(n) &:= \sum_{m<\frac{n}{4}} \sigma_e(m)\sigma_f(n-4m). \end{aligned}$$

Ramanujan [10] and Lahiri [9], [11] showed that $I_{e,f}(n)$ can be expressed as :

$$
\begin{aligned}
I_{1,1}(n) &= \frac{5}{12}\sigma_3(n) + \frac{(1-6n)}{12}\sigma_1(n),\\
I_{1,3}(n) &= \frac{7}{80}\sigma_5(n) + \frac{(1-3n)}{24}\sigma_3(n) - \frac{1}{240}\sigma_1(n),\\
I_{1,5}(n) &= \frac{5}{126}\sigma_7(n) + \frac{(1-2n)}{24}\sigma_5(n) + \frac{1}{504}\sigma_1(n),\\
I_{3,3}(n) &= \frac{1}{120}\sigma_7(n) - \frac{1}{120}\sigma_3(n),\\
I_{1,7}(n) &= \frac{11}{480}\sigma_9(n) + \frac{(2-3n)}{48}\sigma_7(n) - \frac{1}{480}\sigma_1(n),\\
I_{3,5}(n) &= \frac{11}{5040}\sigma_9(n) - \frac{1}{240}\sigma_5(n) + \frac{1}{504}\sigma_3(n),\\
I_{1,9}(n) &= \frac{455}{30404}\sigma_{11}(n) + \frac{(5-6n)}{120}\sigma_9(n) + \frac{1}{264}\sigma_1(n) - \frac{36}{3455}\tau(n),\\
I_{3,7}(n) &= \frac{91}{110560}\sigma_{11}(n) - \frac{1}{240}\sigma_7(n) - \frac{1}{480}\sigma_3(n) + \frac{15}{2764}\tau(n),\\
I_{5,5}(n) &= \frac{65}{174132}\sigma_{11}(n) + \frac{1}{252}\sigma_5(n) - \frac{3}{691}\tau(n),\\
I_{1,11}(n) &= \frac{691}{65520}\sigma_{13}(n) + \frac{(1-n)}{24}\sigma_{11}(n) - \frac{691}{65520}\tau(n),\\
I_{3,9}(n) &= \frac{1}{2640}\sigma_{13}(n) - \frac{1}{240}\sigma_9(n) + \frac{1}{264}\sigma_3(n),\\
I_{5,7}(n) &= \frac{1}{10080}\sigma_{13}(n) + \frac{1}{504}\sigma_7(n) - \frac{1}{480}\sigma_5(n).
\end{aligned}
$$

And in ([6], p. 45-54) we can see that

$$
\begin{aligned}
T_{1,1}(n) &= \frac{1}{12}\sigma_3(n) + \frac{1}{3}\sigma_3(\frac{n}{2}) + \frac{(1-3n)}{24}\sigma_1(n) + \frac{(1-6n)}{24}\sigma_1(\frac{n}{2}),\\
T_{1,3}(n) &= \frac{1}{48}\sigma_5(n) + \frac{1}{15}\sigma_5(\frac{n}{2}) + \frac{(2-3n)}{48}\sigma_3(n) - \frac{1}{240}\sigma_1(\frac{n}{2}),\\
T_{3,1}(n) &= \frac{1}{240}\sigma_5(n) + \frac{1}{12}\sigma_5(\frac{n}{2}) + \frac{(1-3n)}{24}\sigma_3(\frac{n}{2}) - \frac{1}{240}\sigma_1(n),\\
T_{1,5}(n) &= \frac{1}{102}\sigma_7(n) + \frac{32}{1071}\sigma_7(\frac{n}{2}) + \frac{(1-n)}{24}\sigma_5(n) + \frac{1}{504}\sigma_1(\frac{n}{2}) - \frac{1}{102}b(n),\\
T_{3,3}(n) &= \frac{1}{2040}\sigma_7(n) + \frac{2}{255}\sigma_7(\frac{n}{2}) - \frac{1}{240}\sigma_3(n) - \frac{1}{240}\sigma_3(\frac{n}{2}) + \frac{1}{272}b(n),\\
T_{5,1}(n) &= \frac{1}{2142}\sigma_7(n) + \frac{2}{51}\sigma_7(\frac{n}{2}) + \frac{(1-2n)}{24}\sigma_5(\frac{n}{2}) + \frac{1}{504}\sigma_1(n) - \frac{1}{408}b(n),\\
T_{1,7}(n) &= \frac{17}{2976}\sigma_9(n) + \frac{8}{465}\sigma_9(\frac{n}{2}) + \frac{(4-3n)}{96}\sigma_7(n) - \frac{1}{480}\sigma_1(\frac{n}{2})\\
&\quad - \frac{1}{62}c(n) - \frac{16}{31}d(n),\\
T_{3,5}(n) &= \frac{1}{7440}\sigma_9(n) + \frac{4}{1953}\sigma_9(\frac{n}{2}) - \frac{1}{240}\sigma_5(n) + \frac{1}{504}\sigma_3(\frac{n}{2})\\
&\quad + \frac{1}{248}c(n) + \frac{4}{31}d(n),\\
T_{5,3}(n) &= \frac{1}{31248}\sigma_9(n) + \frac{1}{465}\sigma_9(\frac{n}{2}) - \frac{1}{240}\sigma_5(\frac{n}{2}) + \frac{1}{504}\sigma_3(n)\\
&\quad - \frac{1}{496}c(n) - \frac{2}{31}d(n),\\
T_{7,1}(n) &= \frac{1}{14880}\sigma_9(n) + \frac{17}{744}\sigma_9(\frac{n}{2}) + \frac{(2-3n)}{48}\sigma_7(\frac{n}{2}) - \frac{1}{480}\sigma_1(n)\\
&\quad + \frac{1}{496}c(n) + \frac{2}{31}d(n),\\
T_{1,9}(n) &= \frac{31}{8292}\sigma_{11}(n) + \frac{256}{22803}\sigma_{11}(\frac{n}{2}) + \frac{(5-3n)}{120}\sigma_9(n) + \frac{1}{264}\sigma_1(\frac{n}{2})\\
&\quad - \frac{141}{6910}\tau(n) - \frac{2688}{691}\tau(\frac{n}{2}),\\
T_{3,7}(n) &= \frac{17}{331680}\sigma_{11}(n) + \frac{8}{10365}\sigma_{11}(\frac{n}{2}) - \frac{1}{240}\sigma_7(n) - \frac{1}{480}\sigma_3(\frac{n}{2})\\
&\quad + \frac{91}{22112}\tau(n) + \frac{368}{691}\tau(\frac{n}{2}),\\
T_{5,5}(n) &= \frac{1}{174132}\sigma_{11}(n) + \frac{16}{43533}\sigma_{11}(\frac{n}{2}) + \frac{1}{504}\sigma_5(n) + \frac{1}{504}\sigma_5(\frac{n}{2})\\
&\quad - \frac{11}{5528}\tau(n) - \frac{88}{691}\tau(\frac{n}{2}),\\
T_{7,3}(n) &= \frac{1}{331680}\sigma_{11}(n) + \frac{17}{20730}\sigma_{11}(\frac{n}{2}) - \frac{1}{240}\sigma_7(\frac{n}{2}) - \frac{1}{480}\sigma_3(n)\\
&\quad + \frac{23}{11056}\tau(n) + \frac{91}{1382}\tau(\frac{n}{2}),\\
T_{9,1}(n) &= \frac{1}{91212}\sigma_{11}(n) + \frac{31}{2073}\sigma_{11}(\frac{n}{2}) + \frac{(5-6n)}{120}\sigma_9(\frac{n}{2}) + \frac{1}{264}\sigma_1(n)\\
&\quad - \frac{21}{5528}\tau(n) - \frac{282}{3455}\tau(\frac{n}{2}).
\end{aligned}
$$

Also we can find

$$T_{m,3,1}(n) = \frac{1}{720}\left[n\left\{\sigma_5(n) + 20\sigma_5(\frac{n}{2}) - (36n-15)\,\sigma_3(\frac{n}{2})\right\} - b(n)\right] \tag{2.8}$$

in ([12], Theorem 1.1(b)). Moreover, in ([6], p. 45-54) we can observe that

$$\begin{aligned}
U_{1,1}(n) &= \frac{1}{48}\sigma_3(n) + \frac{1}{16}\sigma_3(\frac{n}{2}) + \frac{1}{3}\sigma_3(\frac{n}{4}) + \frac{(2-3n)}{48}\sigma_1(n) + \frac{(1-6n)}{24}\sigma_1(\frac{n}{4}),\\
U_{1,3}(n) &= \frac{1}{192}\sigma_5(n) + \frac{1}{64}\sigma_5(\frac{n}{2}) + \frac{1}{15}\sigma_5(\frac{n}{4}) + \frac{(4-3n)}{96}\sigma_3(n) - \frac{1}{240}\sigma_1(\frac{n}{4}) - \frac{1}{64}a(n),\\
U_{3,1}(n) &= \frac{1}{3840}\sigma_5(n) + \frac{1}{256}\sigma_5(\frac{n}{2}) + \frac{1}{12}\sigma_5(\frac{n}{4}) + \frac{(1-3n)}{24}\sigma_3(\frac{n}{4}) - \frac{1}{240}\sigma_1(n)\\
&\quad + \frac{1}{256}a(n),\\
U_{1,5}(n) &= \frac{1}{408}\sigma_7(n) + \frac{1}{136}\sigma_7(\frac{n}{2}) + \frac{32}{1071}\sigma_7(\frac{n}{4}) + \frac{(2-n)}{48}\sigma_5(n) + \frac{1}{504}\sigma_1(\frac{n}{4})\\
&\quad - \frac{19}{816}b(n) - \frac{26}{51}b(\frac{n}{2}),\\
U_{3,3}(n) &= \frac{1}{32640}\sigma_7(n) + \frac{1}{2176}\sigma_7(\frac{n}{2}) + \frac{2}{255}\sigma_7(\frac{n}{4}) - \frac{1}{240}\sigma_3(n) - \frac{1}{240}\sigma_3(\frac{n}{4})\\
&\quad + \frac{9}{2176}b(n) + \frac{9}{136}b(\frac{n}{2}),\\
U_{5,1}(n) &= \frac{1}{137088}\sigma_7(n) + \frac{1}{2176}\sigma_7(\frac{n}{2}) + \frac{2}{51}\sigma_7(\frac{n}{4}) + \frac{(1-2n)}{24}\sigma_5(\frac{n}{4}) + \frac{1}{504}\sigma_1(n)\\
&\quad - \frac{13}{6528}b(n) - \frac{19}{816}b(\frac{n}{2}),\\
U_{1,9}(n) &= -\frac{7}{16584}\sigma_{11}(n) + \frac{23}{5528}\sigma_{11}(\frac{n}{2}) + \frac{256}{22803}\sigma_{11}(\frac{n}{4}) + \frac{(10-3n)}{240}\sigma_9(n)\\
&\quad + \frac{1}{264}\sigma_1(\frac{n}{4}) - \frac{1589}{55280}\tau(n) - \frac{5790}{691}\tau(\frac{n}{2}) + \frac{2562304}{691}\tau(\frac{n}{4}) + 61440f(n),\\
U_{3,7}(n) &= \frac{121}{2653440}\sigma_{11}(n) + \frac{1}{176896}\sigma_{11}(\frac{n}{2}) + \frac{8}{10365}\sigma_{11}(\frac{n}{4}) - \frac{1}{240}\sigma_7(n)\\
&\quad - \frac{1}{480}\sigma_3(\frac{n}{4}) + \frac{729}{176896}\tau(n) + \frac{6003}{11056}\tau(\frac{n}{2}) - \frac{71496}{691}\tau(\frac{n}{4}) - 1920f(n),\\
U_{5,5}(n) &= \frac{1}{11144448}\sigma_{11}(n) + \frac{1}{176896}\sigma_{11}(\frac{n}{2}) + \frac{16}{43533}\sigma_{11}(\frac{n}{4}) + \frac{1}{504}\sigma_5(n)\\
&\quad + \frac{1}{504}\sigma_5(\frac{n}{4}) - \frac{351}{176896}\tau(n) - \frac{2505}{22112}\tau(\frac{n}{2}) - \frac{5616}{691}\tau(\frac{n}{4}),\\
U_{9,1}(n) &= \frac{31}{5837568}\sigma_{11}(n) + \frac{1}{176896}\sigma_{11}(\frac{n}{2}) + \frac{31}{2073}\sigma_{11}(\frac{n}{4}) + \frac{(5-6n)}{120}\sigma_9(\frac{n}{4})\\
&\quad + \frac{1}{264}\sigma_1(n) - \frac{671}{176896}\tau(n) - \frac{2505}{22112}\tau(\frac{n}{2}) - \frac{105314}{3455}\tau(\frac{n}{4}) - 240f(n)
\end{aligned}$$

and from ([13], (2.8), (3.1)) we know that

$$U_{1,7}(n) = \frac{17}{11904}\sigma_9(n) + \frac{17}{3968}\sigma_9(\frac{n}{2}) + \frac{8}{465}\sigma_9(\frac{n}{4}) + \frac{(8-3n)}{192}\sigma_7(n) - \frac{1}{480}\sigma_1(\frac{n}{4})$$
$$+ \frac{433}{248}d(n) - \frac{4232}{31}d(\frac{n}{2}) - \frac{109}{3968}c(n) - \frac{529}{124}c(\frac{n}{2}),$$
$$U_{3,5}(n) = \frac{1}{119040}\sigma_9(n) + \frac{1}{7936}\sigma_9(\frac{n}{2}) + \frac{4}{1953}\sigma_9(\frac{n}{4}) - \frac{1}{240}\sigma_5(n) + \frac{1}{504}\sigma_3(\frac{n}{4})$$
$$- \frac{27}{496}d(n) + \frac{252}{31}d(\frac{n}{2}) + \frac{33}{7936}c(n) + \frac{63}{248}c(\frac{n}{2}),$$
$$U_{5,3}(n) = \frac{1}{1999872}\sigma_9(n) + \frac{1}{31744}\sigma_9(\frac{n}{2}) + \frac{1}{465}\sigma_9(\frac{n}{4}) - \frac{1}{240}\sigma_5(\frac{n}{4}) + \frac{1}{504}\sigma_3(n)$$
$$- \frac{33}{1984}d(n) - \frac{33}{31}d(\frac{n}{2}) - \frac{63}{31744}c(n) - \frac{33}{992}c(\frac{n}{2}),$$
$$U_{7,1}(n) = \frac{1}{3809280}\sigma_9(n) + \frac{17}{253952}\sigma_9(\frac{n}{2}) + \frac{17}{744}\sigma_9(\frac{n}{4}) + \frac{(2-3n)}{48}\sigma_7(\frac{n}{4})$$
$$- \frac{1}{480}\sigma_1(n) + \frac{407}{15872}d(n) + \frac{109}{248}d(\frac{n}{2}) + \frac{529}{253952}c(n) + \frac{109}{7936}c(\frac{n}{2}),$$
$$U_{7,3}(n) = -\frac{7}{2653440}\sigma_{11}(n) + \frac{1}{176896}\sigma_{11}(\frac{n}{2}) + \frac{17}{20730}\sigma_{11}(\frac{n}{4}) - \frac{1}{240}\sigma_7(\frac{n}{4})$$
$$- \frac{1}{480}\sigma_3(n) + \frac{369}{176896}\tau(n) + \frac{1641}{22112}\tau(\frac{n}{2}) + \frac{22203}{1382}\tau(\frac{n}{4}) + 120f(n).$$

Many identities in Proposition 2.2 are found in the wide area of [6].

Proposition 2.2. *(See [6]), For $q \in \mathbb{C}$ with $|q| < 1$, we have*

(a)
$$L(q)M(q^2) = 2L(q^2)M(q^2) + \frac{1}{21}N(q) - \frac{22}{21}N(q^2),$$

(b)
$$M(q)L(q^2) = \frac{1}{2}L(q)M(q) - \frac{11}{42}N(q) + \frac{16}{21}N(q^2),$$

(c)
$$L(q)M(q^4) = 4L(q^4)M(q^4) + \frac{1}{336}N(q) + \frac{5}{112}N(q^2) - \frac{64}{21}N(q^4) - \frac{45}{2}A(q),$$

(d)
$$L(q^4)M(q) = \frac{1}{4}L(q)M(q) - \frac{4}{21}N(q) + \frac{5}{28}N(q^2) + \frac{16}{21}N(q^4) + 90A(q),$$

(e)
$$M(q)M(q^2) = \frac{1}{17}M^2(q) + \frac{16}{17}M^2(q^2) + \frac{3600}{17}B(q),$$

(f)
$$L(q)N(q^2) = 2L(q^2)N(q^2) + \frac{1}{85}M^2(q) - \frac{86}{85}M^2(q^2) - \frac{504}{17}B(q),$$

(g)
$$N(q)L(q^2) = \frac{1}{2}L(q)N(q) - \frac{43}{170}M^2(q) + \frac{64}{85}M^2(q^2) - \frac{2016}{17}B(q),$$

(h)

$$L(q)N(q^4) = 4L(q^4)N(q^4) + \frac{1}{5440}M^2(q) + \frac{63}{5440}M^2(q^2) - \frac{256}{85}M^2(q^4) - \frac{819}{34}B(q) - \frac{4788}{17}B(q^2),$$

(i)

$$M(q)M(q^4) = \frac{1}{272}M^2(q) + \frac{15}{272}M^2(q^2) + \frac{16}{17}M^2(q^4) + \frac{4050}{17}B(q) + \frac{64800}{17}B(q^2),$$

(j)

$$L(q^4)N(q) = \frac{1}{4}L(q)N(q) - \frac{16}{85}M^2(q) + \frac{63}{340}M^2(q^2) + \frac{64}{85}M^2(q^4) - \frac{4788}{17}B(q) - \frac{104832}{17}B(q^2),$$

(k)

$$L(q)M^2(q^2) = 2L(q^2)M^2(q^2) + \frac{1}{341}M(q)N(q) - \frac{342}{341}M(q^2)N(q^2) - \frac{720}{31}C(q) - \frac{23040}{31}D(q),$$

(l)

$$M^2(q)L(q^2) = \frac{1}{2}L(q)M^2(q) - \frac{171}{682}M(q)N(q) + \frac{256}{341}M(q^2)N(q^2) + \frac{5760}{31}C(q) + \frac{184320}{31}D(q).$$

And we obtained more simplified identities as follows in ([8], (22))

$$\begin{aligned}
L(q)M^2(q^4) &= 4L(q^4)M^2(q^4) + \frac{1}{87296}M(q)N(q) + \frac{255}{87296}M(q^2)N(q^2) \\
&\quad - \frac{1024}{341}M(q^4)N(q^4) - \frac{23805}{992}C(q) - \frac{4905}{31}C(q^2) - \frac{18315}{62}D(q) - \frac{156960}{31}D(q^2), \\
M(q)N(q^4) &= \frac{5}{21824}M(q)N(q) + \frac{315}{21824}M(q^2)N(q^2) + \frac{336}{341}M(q^4)N(q^4) \\
&\quad + \frac{59535}{248}C(q) + \frac{124740}{31}C(q^2) + \frac{62370}{31}D(q) + \frac{3991680}{31}D(q^2), \\
N(q)M(q^4) &= \frac{21}{5456}M(q)N(q) + \frac{315}{5456}M(q^2)N(q^2) + \frac{320}{341}M(q^4)N(q^4) \\
&\quad - \frac{31185}{62}C(q) - \frac{952560}{31}C(q^2) + \frac{204120}{31}D(q) - \frac{30481920}{31}D(q^2), \\
M^2(q)L(q^4) &= \frac{1}{4}L(q)M^2(q) - \frac{64}{341}M(q)N(q) + \frac{255}{1364}M(q^2)N(q^2) \\
&\quad + \frac{256}{341}M(q^4)N(q^4) + \frac{9810}{31}C(q) + \frac{1523520}{31}C(q^2) - \frac{623520}{31}D(q) \\
&\quad + \frac{48752640}{31}D(q^2)
\end{aligned} \tag{2.9}$$

owing to N. Cheng and K. S. Williams' results in ([6], Theorem 6.1).

Proposition 2.3. *For $q \in \mathbb{C}$ with $|q| < 1$, we have*

(a) *(See ([8], Theorem 2.5(g)))*

$$L(q)A(q) = -\sum_{n=1}^{\infty} b(n)q^n - 32\sum_{n=1}^{\infty} b(n)q^{2n} + 2\sum_{n=1}^{\infty} na(n)q^n,$$

(b) *(See ([8], Theorem 2.5(h)))*

$$L(q)B(q^2) = 3\sum_{n=1}^{\infty} d(n)q^n - 160\sum_{n=1}^{\infty} d(n)q^{2n} - 5\sum_{n=1}^{\infty} c(n)q^{2n} + 3\sum_{n=1}^{\infty} nb(n)q^{2n},$$

(c) *(See ([8], Theorem 2.5(i)))*

$$M(q)A(q) = -256\sum_{n=1}^{\infty} d(n)q^n + 16384\sum_{n=1}^{\infty} d(n)q^{2n} + \sum_{n=1}^{\infty} c(n)q^n + 512\sum_{n=1}^{\infty} c(n)q^{2n},$$

(d) *(See ([8], Theorem 2.5(j)))*

$$M(q^2)A(q) = -16\sum_{n=1}^{\infty} d(n)q^n + 1024\sum_{n=1}^{\infty} d(n)q^{2n} + \sum_{n=1}^{\infty} c(n)q^n + 32\sum_{n=1}^{\infty} c(n)q^{2n},$$

(e) *(See ([12], Theorem 1.2 (a)))*

$$\begin{aligned} L(q)L(q^2)M(q^2) = 1 &+ \frac{96}{17}\sum_{n=1}^{\infty} \sigma_7(n)q^n + \frac{8064}{17}\sum_{n=1}^{\infty} \sigma_7(n)q^{2n} - 24\sum_{n=1}^{\infty} n\sigma_5(n)q^n \\ &- 2976\sum_{n=1}^{\infty} n\sigma_5(n)q^{2n} + 3456\sum_{n=1}^{\infty} n^2\sigma_3(n)q^{2n} - \frac{96}{17}\sum_{n=1}^{\infty} b(n)q^n, \end{aligned}$$

(f) *(See ([12], Theorem 1.2 (j)))*

$$L(q)B(q) = -16\sum_{n=1}^{\infty} d(n)q^n - \frac{1}{2}\sum_{n=1}^{\infty} c(n)q^n + \frac{3}{2}\sum_{n=1}^{\infty} nb(n)q^n,$$

(g) *(See ([12], Theorem 1.2 (k)))*

$$L(q^2)B(q) = 8\sum_{n=1}^{\infty} d(n)q^n + \frac{1}{4}\sum_{n=1}^{\infty} c(n)q^n + \frac{3}{4}\sum_{n=1}^{\infty} nb(n)q^n,$$

(h) *(See ([12], Theorem 2.1))*

$$\begin{aligned} L(q)L(q^2)M(q) = 1 &+ \frac{2016}{17}\sum_{n=1}^{\infty} \sigma_7(n)q^n + \frac{6144}{17}\sum_{n=1}^{\infty} \sigma_7(n)q^{2n} - 744\sum_{n=1}^{\infty} n\sigma_5(n)q^n \\ &- 1536\sum_{n=1}^{\infty} n\sigma_5(n)q^{2n} + 864\sum_{n=1}^{\infty} n^2\sigma_3(n)q^n - \frac{384}{17}\sum_{n=1}^{\infty} b(n)q^n, \end{aligned}$$

(i) *(See ([13], Theorem 1.3 (d)))*

$$\begin{aligned} L(q)L(q^4)M(q) = 1 &+ \frac{504}{17}\sum_{n=1}^{\infty} \sigma_7(n)q^n + \frac{1512}{17}\sum_{n=1}^{\infty} \sigma_7(n)q^{2n} + \frac{6144}{17}\sum_{n=1}^{\infty} \sigma_7(n)q^{4n} \\ &- 312\sum_{n=1}^{\infty} n\sigma_5(n)q^n - 360\sum_{n=1}^{\infty} n\sigma_5(n)q^{2n} - 3072\sum_{n=1}^{\infty} n\sigma_5(n)q^{4n} \\ &+ 432\sum_{n=1}^{\infty} n^2\sigma_3(n)q^n - \frac{1932}{17}\sum_{n=1}^{\infty} b(n)q^n - \frac{52608}{17}\sum_{n=1}^{\infty} b(n)q^{2n} \\ &+ 180\sum_{n=1}^{\infty} na(n)q^n, \end{aligned}$$

(j) *(See ([13], Theorem 1.3 (e)))*

$$\begin{aligned}L^2(q^4)M(q) = 1 &+ \frac{30}{17}\sum_{n=1}^{\infty}\sigma_7(n)q^n + \frac{450}{17}\sum_{n=1}^{\infty}\sigma_7(n)q^{2n} + \frac{7680}{17}\sum_{n=1}^{\infty}\sigma_7(n)q^{4n}\\ &- 30\sum_{n=1}^{\infty}n\sigma_5(n)q^n - 180\sum_{n=1}^{\infty}n\sigma_5(n)q^{2n} - 1536\sum_{n=1}^{\infty}n\sigma_5(n)q^{4n}\\ &+ 108\sum_{n=1}^{\infty}n^2\sigma_3(n)q^n + \frac{1194}{17}\sum_{n=1}^{\infty}b(n)q^n + \frac{12576}{17}\sum_{n=1}^{\infty}b(n)q^{2n}\\ &+ 90\sum_{n=1}^{\infty}na(n)q^n,\end{aligned}$$

(k) *(See ([13], Theorem 1.3 (f)))*

$$\begin{aligned}&L(q)L(q^4)M(q^4)\\ &= 1 + \frac{3}{34}\sum_{n=1}^{\infty}\sigma_7(n)q^n + \frac{189}{34}\sum_{n=1}^{\infty}\sigma_7(n)q^{2n} + \frac{8064}{17}\sum_{n=1}^{\infty}\sigma_7(n)q^{4n} - \frac{3}{4}\sum_{n=1}^{\infty}n\sigma_5(n)q^n\\ &\quad - \frac{45}{2}\sum_{n=1}^{\infty}n\sigma_5(n)q^{2n} - 4992\sum_{n=1}^{\infty}n\sigma_5(n)q^{4n} + 6912\sum_{n=1}^{\infty}n^2\sigma_3(n)q^{4n}\\ &\quad - \frac{411}{34}\sum_{n=1}^{\infty}b(n)q^n - \frac{1932}{17}\sum_{n=1}^{\infty}b(n)q^{2n} - \frac{45}{4}\sum_{n=1}^{\infty}na(n)q^n,\end{aligned}$$

(l) *(See ([13], (2.11)))*

$$L(q^4)B(q) = 8\sum_{n=1}^{\infty}d(n)q^n + \frac{5}{8}\sum_{n=1}^{\infty}c(n)q^n + \frac{3}{8}\sum_{n=1}^{\infty}nb(n)q^n.$$

In [8], [12], and [7] we induced various convolution sum formulae which are the base of obtaining another convolution sums.

Proposition 2.4. *Let $n \in \mathbb{N}$. Then we have*

(a) *(See ([8], Lemma 1.5(d)))*

$$\sum_{m<\frac{n}{2}} b(m)\sigma_1(n-2m) = -\frac{1}{48}\left\{6d(n) - 320d(\frac{n}{2}) - 10c(\frac{n}{2}) + (3n-2)\,b(\frac{n}{2})\right\},$$

(b) *(See ([12], Lemma 3.1(a)))*

$$\sum_{m=1}^{n-1}\sigma_1(m)b(n-m) = \frac{1}{48}\left\{32d(n) + c(n) - (3n-2)\,b(n)\right\},$$

(c) *(See ([12], Lemma 3.1(b)))*

$$\sum_{m<\frac{n}{2}}\sigma_1(m)b(n-2m) = -\frac{1}{96}\left\{32d(n) + c(n) + (3n-4)\,b(n)\right\},$$

(d) *(See ([12], Lemma 3.1(c)))*

$$\sum_{m=1}^{n-1}\sigma_3(m)b(n-m) = \frac{1}{240}\left\{\tau(n) + 256\tau(\frac{n}{2}) - b(n)\right\},$$

(e) *(See ([7], (3.5)))*

$$\sum_{m<\frac{n}{2}} \sigma_3(m)b(n-2m) = \frac{1}{240}\left\{\tau(n) + 16\tau(\frac{n}{2}) - b(n)\right\},$$

(f) *(See ([7], (3.7)))*

$$\sum_{m<\frac{n}{2}} b(m)\sigma_3(n-2m) = \frac{1}{2653440}\Big\{15\sigma_{11}(n) - 15\sigma_{11}(\frac{n}{2}) - 15\tau(n) - 20024\tau(\frac{n}{2})$$
$$-39624704\tau(\frac{n}{4}) - 679280640f(n) - 11056b(\frac{n}{2})\Big\}.$$

Proposition 2.5. *(See ([1], Lemma 2.2, Lemma 2.3)) For $q \in \mathbb{C}$ with $|q| < 1$, we have*

(a)

$$xw^8 = \frac{1}{255}M^2(q) - \frac{1}{255}M^2(q^2) + \frac{240}{17}B(q) + 256B(q^2),$$

(b)

$$x^2w^8 = \frac{1}{255}M^2(q) - \frac{1}{255}M^2(q^2) - \frac{32}{17}B(q),$$

(c)

$$x^3w^8 = \frac{1}{255}M^2(q) - \frac{1}{255}M^2(q^2) - \frac{32}{17}B(q) - 256B(q^2),$$

(d)

$$x^2\sqrt{1-x}w^8 = 8192B(q^4) + 256B(q^2) + \frac{1}{2}H(q).$$

3 Preparations to Find $\sum_{m<\frac{n}{8}} \sigma_1(m)\sigma_5(n-8m)$ and $\sum_{m<\frac{n}{8}} \sigma_5(m)\sigma_1(n-8m)$

Proposition 2.5 enables us to induce Corollary 3.1 :

Corollary 3.1. *For $q \in \mathbb{C}$ with $|q| < 1$, we obtain*

(a)

$$x^3w^{10} = \frac{8}{31}\sum_{n=1}^{\infty}\sigma_9(n)q^n - \frac{8}{31}\sum_{n=1}^{\infty}\sigma_9(n)q^{2n} + \frac{3712}{31}\sum_{n=1}^{\infty}d(n)q^n - 8192\sum_{n=1}^{\infty}d(n)q^{2n}$$
$$- \frac{8}{31}\sum_{n=1}^{\infty}c(n)q^n - 256\sum_{n=1}^{\infty}c(n)q^{2n},$$

(b)

$$x^4w^{10} = \frac{8}{31}\sum_{n=1}^{\infty}\sigma_9(n)q^n - \frac{8}{31}\sum_{n=1}^{\infty}\sigma_9(n)q^{2n} + \frac{19584}{31}\sum_{n=1}^{\infty}d(n)q^n$$
$$- 24576\sum_{n=1}^{\infty}d(n)q^{2n} - \frac{8}{31}\sum_{n=1}^{\infty}c(n)q^n - 768\sum_{n=1}^{\infty}c(n)q^{2n}.$$

Proof. (a) First by (1.7) and (1.13) we can observe that

$$\begin{aligned}
&4L(q^4) - L(q)\\
&= 4\left\{(1-\frac{5}{4}x)w^2 + 3x(1-x)w\frac{dw}{dx}\right\} - \left\{(1-5x)w^2 + 12x(1-x)w\frac{dw}{dx}\right\}\\
&= 3w^2,
\end{aligned}$$

which shows that

$$w^2 = \frac{4}{3}L(q^4) - \frac{1}{3}L(q). \tag{3.1}$$

Therefore by Proposition 2.5 (c) and (3.1) we have

$$\begin{aligned}
x^3w^{10} &= x^3w^8 \cdot w^2\\
&= \left(\frac{1}{255}M^2(q) - \frac{1}{255}M^2(q^2) - \frac{32}{17}B(q) - 256B(q^2)\right)\left(\frac{4}{3}L(q^4) - \frac{1}{3}L(q)\right)\\
&= \frac{4}{765}M^2(q)L(q^4) - \frac{1}{765}M^2(q)L(q) - \frac{4}{765}M^2(q^2)L(q^4) + \frac{1}{765}M^2(q^2)L(q)\\
&\quad - \frac{128}{51}B(q)L(q^4) + \frac{32}{51}B(q)L(q) - \frac{1024}{3}B(q^2)L(q^4) + \frac{256}{3}B(q^2)L(q)
\end{aligned}$$

so we refer to (2.7), Proposition 2.2 (k), (l), (2.9), Proposition 2.3 (b), (f), (g), and (l).

(b) From (1.7) and (1.10) we induce that

$$\begin{aligned}
&2L(q^2) - L(q)\\
&= 2\left\{(1-2x)w^2 + 6x(1-x)w\frac{dw}{dx}\right\} - \left\{(1-5x)w^2 + 12x(1-x)w\frac{dw}{dx}\right\}\\
&= w^2 + xw^2\\
&= \frac{4}{3}L(q^4) - \frac{1}{3}L(q) + xw^2,
\end{aligned}$$

where we use (3.1) for the last line and so we obtain

$$xw^2 = -\frac{4}{3}L(q^4) + 2L(q^2) - \frac{2}{3}L(q). \tag{3.2}$$

Thus by Proposition 2.5 (c) and (3.2) we have

$$\begin{aligned}
x^4w^{10} &= x^3w^8 \cdot xw^2\\
&= \left(\frac{1}{255}M^2(q) - \frac{1}{255}M^2(q^2) - \frac{32}{17}B(q) - 256B(q^2)\right)\\
&\quad \times \left(-\frac{4}{3}L(q^4) + 2L(q^2) - \frac{2}{3}L(q)\right)\\
&= -\frac{4}{765}M^2(q)L(q^4) + \frac{2}{255}M^2(q)L(q^2) - \frac{2}{765}M^2(q)L(q) + \frac{4}{765}M^2(q^2)L(q^4)\\
&\quad - \frac{2}{255}M^2(q^2)L(q^2) + \frac{2}{765}M^2(q^2)L(q) + \frac{128}{51}B(q)L(q^4) - \frac{64}{17}B(q)L(q^2)\\
&\quad + \frac{64}{51}B(q)L(q) + \frac{1024}{3}B(q^2)L(q^4) - 512B(q^2)L(q^2) + \frac{512}{3}B(q^2)L(q)
\end{aligned}$$

so we refer to (2.7), Proposition 2.2 (k), (l), (2.9), Proposition 2.3 (b), (f), (g), and (l).

□

In ([14], (3), (30)) K. S. Williams defined

$$K(q) := \sum_{n=1}^{\infty} k(n)q^n = q\prod_{n=1}^{\infty}(1-q^{2n})^4(1-q^{4n})^4 = \frac{1}{16}x\sqrt{1-x}w^4 \tag{3.3}$$

and evaluated

$$\big(L(q)-8L(q^8)\big)^2 = \frac{4}{5}M(q) - \frac{3}{5}M(q^2) - \frac{12}{5}M(q^4) + \frac{256}{5}M(q^8) + 144K(q). \tag{3.4}$$

Remark 3.1. Let us expand Eq. (3.4) to obtain $L(q)L(q^8)$:

$$\begin{aligned}&\big(L(q)-8L(q^8)\big)^2\\ &= L^2(q) - 16L(q)L(q^8) + 64L^2(q^8)\\ &= \frac{4}{5}M(q) - \frac{3}{5}M(q^2) - \frac{12}{5}M(q^4) + \frac{256}{5}M(q^8) + 144K(q)\end{aligned}$$

and so

$$L(q)L(q^8) = \frac{1}{16}L^2(q) + 4L^2(q^8) - \frac{1}{20}M(q) + \frac{3}{80}M(q^2) + \frac{3}{20}M(q^4) - \frac{16}{5}M(q^8) - 9K(q). \tag{3.5}$$

Theorem 3.2. *For $q \in \mathbb{C}$ with $|q| < 1$, we obtain*

(a)

$$M(q)K(q) = \frac{1}{2}\sum_{n=1}^{\infty} h(n)q^n + \frac{1}{16}\sum_{n=1}^{\infty} g(n)q^n + 224\sum_{n=1}^{\infty} b(n)q^{2n} + 7168\sum_{n=1}^{\infty} b(n)q^{4n},$$

(b)

$$M(q^2)K(q) = \frac{1}{32}\sum_{n=1}^{\infty} h(n)q^n + \frac{1}{16}\sum_{n=1}^{\infty} g(n)q^n - 16\sum_{n=1}^{\infty} b(n)q^{2n} - 512\sum_{n=1}^{\infty} b(n)q^{4n},$$

(c)

$$M(q^4)K(q) = -\frac{7}{256}\sum_{n=1}^{\infty} h(n)q^n + \frac{1}{16}\sum_{n=1}^{\infty} g(n)q^n - 16\sum_{n=1}^{\infty} b(n)q^{2n} - 512\sum_{n=1}^{\infty} b(n)q^{4n},$$

(d)

$$\begin{aligned}M(q^8)K(q) = &-\frac{67}{4096}\sum_{n=1}^{\infty} h(n)q^n + \frac{17}{512}\sum_{n=1}^{\infty} g(n)q^n + \frac{15}{32}\sum_{n=1}^{\infty} b(n)q^n\\ &- \frac{19}{4}\sum_{n=1}^{\infty} b(n)q^{2n} - 272\sum_{n=1}^{\infty} b(n)q^{4n},\end{aligned}$$

(e)

$$\begin{aligned}L^2(q)M(q^2) = 1 &+ \frac{480}{17}\sum_{n=1}^{\infty} \sigma_7(n)q^n + \frac{7680}{17}\sum_{n=1}^{\infty} \sigma_7(n)q^{2n} - 96\sum_{n=1}^{\infty} n\sigma_5(n)q^n\\ &- 3840\sum_{n=1}^{\infty} n\sigma_5(n)q^{2n} + 6912\sum_{n=1}^{\infty} n^2\sigma_3(n)q^{2n} + \frac{336}{17}\sum_{n=1}^{\infty} b(n)q^n,\end{aligned}$$

(f)

$$\begin{aligned}
&L(q)L(q^8)M(q^2)\\
&= 1 + \frac{6}{17}\sum_{n=1}^{\infty}\sigma_7(n)q^n + \frac{498}{17}\sum_{n=1}^{\infty}\sigma_7(n)q^{2n} + \frac{1512}{17}\sum_{n=1}^{\infty}\sigma_7(n)q^{4n}\\
&\quad + \frac{6144}{17}\sum_{n=1}^{\infty}\sigma_7(n)q^{8n} - 6\sum_{n=1}^{\infty}n\sigma_5(n)q^n - 360\sum_{n=1}^{\infty}n\sigma_5(n)q^{2n}\\
&\quad - 720\sum_{n=1}^{\infty}n\sigma_5(n)q^{4n} - 6144\sum_{n=1}^{\infty}n\sigma_5(n)q^{8n} + 864\sum_{n=1}^{\infty}n^2\sigma_3(n)q^{2n}\\
&\quad - \frac{9}{32}\sum_{n=1}^{\infty}h(n)q^n - \frac{9}{16}\sum_{n=1}^{\infty}g(n)q^n - \frac{159}{17}\sum_{n=1}^{\infty}b(n)q^n - \frac{5196}{17}\sum_{n=1}^{\infty}b(n)q^{2n}\\
&\quad - \frac{78720}{17}\sum_{n=1}^{\infty}b(n)q^{4n} + 360\sum_{n=1}^{\infty}na(n)q^{2n},
\end{aligned}$$

(g)

$$\begin{aligned}
L(q^8)N(q) &= 1 + \frac{126}{17}\sum_{n=1}^{\infty}\sigma_7(n)q^n + \frac{378}{17}\sum_{n=1}^{\infty}\sigma_7(n)q^{2n} + \frac{1512}{17}\sum_{n=1}^{\infty}\sigma_7(n)q^{4n}\\
&\quad + \frac{6144}{17}\sum_{n=1}^{\infty}\sigma_7(n)q^{8n} - 126\sum_{n=1}^{\infty}n\sigma_5(n)q^n - \frac{189}{32}\sum_{n=1}^{\infty}h(n)q^n\\
&\quad - \frac{189}{16}\sum_{n=1}^{\infty}g(n)q^n - \frac{3339}{17}\sum_{n=1}^{\infty}b(n)q^n - \frac{133308}{17}\sum_{n=1}^{\infty}b(n)q^{2n}\\
&\quad - \frac{1749888}{17}\sum_{n=1}^{\infty}b(n)q^{4n},
\end{aligned}$$

(h)

$$L(q)A(q^2) = -\frac{3}{512}\sum_{n=1}^{\infty}h(n)q^n - 3\sum_{n=1}^{\infty}b(n)q^{2n} - 96\sum_{n=1}^{\infty}b(n)q^{4n} - 4\sum_{n=1}^{\infty}na(n)q^{2n},$$

(i)

$$\begin{aligned}
L(q)N(q^8) &= 1 + \frac{3}{2176}\sum_{n=1}^{\infty}\sigma_7(n)q^n + \frac{189}{2176}\sum_{n=1}^{\infty}\sigma_7(n)q^{2n} + \frac{189}{34}\sum_{n=1}^{\infty}\sigma_7(n)q^{4n}\\
&\quad + \frac{8064}{17}\sum_{n=1}^{\infty}\sigma_7(n)q^{8n} - 8064\sum_{n=1}^{\infty}n\sigma_5(n)q^{8n} + \frac{1323}{4096}\sum_{n=1}^{\infty}h(n)q^n\\
&\quad - \frac{189}{256}\sum_{n=1}^{\infty}g(n)q^n - \frac{26523}{2176}\sum_{n=1}^{\infty}b(n)q^n + \frac{5229}{272}\sum_{n=1}^{\infty}b(n)q^{2n}\\
&\quad + \frac{89460}{17}\sum_{n=1}^{\infty}b(n)q^{4n},
\end{aligned}$$

(j)

$$\begin{aligned}
&L(q)L(q^8)M(q^4)\\
&= 1 + \frac{3}{136}\sum_{n=1}^{\infty}\sigma_7(n)q^n + \frac{189}{136}\sum_{n=1}^{\infty}\sigma_7(n)q^{2n} + \frac{1992}{17}\sum_{n=1}^{\infty}\sigma_7(n)q^{4n}\\
&\quad + \frac{6144}{17}\sum_{n=1}^{\infty}\sigma_7(n)q^{8n} - \frac{3}{8}\sum_{n=1}^{\infty}n\sigma_5(n)q^n - \frac{45}{4}\sum_{n=1}^{\infty}n\sigma_5(n)q^{2n}\\
&\quad - 1440\sum_{n=1}^{\infty}n\sigma_5(n)q^{4n} - 6144\sum_{n=1}^{\infty}n\sigma_5(n)q^{8n} + 3456\sum_{n=1}^{\infty}n^2\sigma_3(n)q^{4n}\\
&\quad + \frac{63}{256}\sum_{n=1}^{\infty}h(n)q^n - \frac{9}{16}\sum_{n=1}^{\infty}g(n)q^n - \frac{1227}{136}\sum_{n=1}^{\infty}b(n)q^n + \frac{537}{17}\sum_{n=1}^{\infty}b(n)q^{2n}\\
&\quad + \frac{68160}{17}\sum_{n=1}^{\infty}b(n)q^{4n} - \frac{45}{8}\sum_{n=1}^{\infty}na(n)q^n,
\end{aligned}$$

(k)

$$\begin{aligned}
L(q^8)A(q) = -\frac{3}{256}\sum_{n=1}^{\infty}h(n)q^n + \frac{3}{128}\sum_{n=1}^{\infty}g(n)q^n + \frac{3}{8}\sum_{n=1}^{\infty}b(n)q^n - 3\sum_{n=1}^{\infty}b(n)q^{2n}\\
- 192\sum_{n=1}^{\infty}b(n)q^{4n} + \frac{1}{4}\sum_{n=1}^{\infty}na(n)q^n,
\end{aligned}$$

(l)

$$A(q)K(q) = \sum_{n=1}^{\infty}c(n)q^{2n},$$

(m)

$$A(q^2)K(q) = -\frac{1}{8}\sum_{n=1}^{\infty}d(n)q^n + 4\sum_{n=1}^{\infty}d(n)q^{2n} + \frac{1}{8}\sum_{n=1}^{\infty}c(n)q^{2n}.$$

Proof. (a) By (1.8) and (3.3) we can know that

$$\begin{aligned}
M(q)K(q) &= (1+14x+x^2)w^4\cdot\frac{1}{16}x\sqrt{1-x}w^4\\
&= \frac{1}{16}\left(x\sqrt{1-x}w^8 + 14x^2\sqrt{1-x}w^8 + x^3\sqrt{1-x}w^8\right)\\
&= \frac{1}{16}\left\{G(q) + 14\left(8192B(q^4) + 256B(q^2) + \frac{1}{2}H(q)\right) + H(q)\right\}\\
&= \frac{1}{16}G(q) + 7168B(q^4) + 224B(q^2) + \frac{1}{2}H(q),
\end{aligned}$$

where we use (1.19) and Proposition 2.5 (d) for the third line.

(b) In a similar manner to proof of Theorem 3.2 (a), by (1.11) and (3.3) we have

$$\begin{aligned} M(q^2)K(q) &= (1-x+x^2)w^4 \cdot \frac{1}{16}x\sqrt{1-x}w^4 \\ &= \frac{1}{16}\left(x\sqrt{1-x}w^8 - x^2\sqrt{1-x}w^8 + x^3\sqrt{1-x}w^8\right) \\ &= \frac{1}{16}\left\{G(q) - \left(8192B(q^4) + 256B(q^2) + \frac{1}{2}H(q)\right) + H(q)\right\} \\ &= \frac{1}{16}G(q) - 512B(q^4) - 16B(q^2) + \frac{1}{32}H(q). \end{aligned}$$

(c) From (1.14) and (3.3) we obtain

$$\begin{aligned} M(q^4)K(q) &= (1-x+\frac{1}{16}x^2)w^4 \cdot \frac{1}{16}x\sqrt{1-x}w^4 \\ &= \frac{1}{16}\left(x\sqrt{1-x}w^8 - x^2\sqrt{1-x}w^8 + \frac{1}{16}x^3\sqrt{1-x}w^8\right) \\ &= \frac{1}{16}\left\{G(q) - \left(8192B(q^4) + 256B(q^2) + \frac{1}{2}H(q)\right) + \frac{1}{16}H(q)\right\} \\ &= \frac{1}{16}G(q) - 512B(q^4) - 16B(q^2) - \frac{7}{256}H(q). \end{aligned}$$

(d) From (1.18) and (3.3) we can induce that

$$\begin{aligned} &M(q^8)K(q) = M(q^8)\cdot K(q) \\ &= \left(-\frac{1}{32}M(q^2) + \frac{9}{16}M(q^4) + \frac{15}{32}\sqrt{1-x}w^4 - \frac{15}{64}x\sqrt{1-x}w^4\right)K(q) \\ &= -\frac{1}{32}M(q^2)K(q) + \frac{9}{16}M(q^4)K(q) + \left(\frac{15}{32}\sqrt{1-x}w^4 - \frac{15}{64}x\sqrt{1-x}w^4\right)K(q) \\ &= -\frac{1}{32}M(q^2)K(q) + \frac{9}{16}M(q^4)K(q) \\ &\quad + \left(\frac{15}{32}\sqrt{1-x}w^4 - \frac{15}{64}x\sqrt{1-x}w^4\right)\cdot\frac{1}{16}x\sqrt{1-x}w^4 \\ &= -\frac{1}{32}M(q^2)K(q) + \frac{9}{16}M(q^4)K(q) + \frac{15}{1024}\left(x^3w^8 - 3x^2w^8 + 2xw^8\right). \end{aligned}$$

Then by Proposition 2.5 (a), (b), and (c) the above equation can be written as

$$\begin{aligned} &M(q^8)K(q) \\ &= -\frac{1}{32}M(q^2)K(q) + \frac{9}{16}M(q^4)K(q) \\ &\quad + \frac{15}{1024}\left\{\left(\frac{1}{255}M^2(q) - \frac{1}{255}M^2(q^2) - \frac{32}{17}B(q) - 256B(q^2)\right)\right. \\ &\quad - 3\left(\frac{1}{255}M^2(q) - \frac{1}{255}M^2(q^2) - \frac{32}{17}B(q)\right) \\ &\quad \left. +2\left(\frac{1}{255}M^2(q) - \frac{1}{255}M^2(q^2) + \frac{240}{17}B(q) + 256B(q^2)\right)\right\} \\ &= -\frac{1}{32}M(q^2)K(q) + \frac{9}{16}M(q^4)K(q) + \frac{15}{32}B(q) + \frac{15}{4}B(q^2) \end{aligned}$$

so we use Theorem 3.2 (b) and (c).

(e) By (1.4) and (2.5) we have

$$
\begin{aligned}
&L^2(q)M(q^2) = L^2(q)\cdot M(q^2)\\
&= \left(1 - 288\sum_{n=1}^{\infty} n\sigma_1(n)q^n + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n\right)\left(1 + 240\sum_{m=1}^{\infty}\sigma_3(m)q^{2m}\right)\\
&= 1 + \sum_{N=1}^{\infty}\Bigg\{-288N\sigma_1(N) + 240\sigma_3(N) + 240\sigma_3(\frac{N}{2})\\
&\qquad - 288\cdot 240\sum_{m<\frac{N}{2}}(N-2m)\,\sigma_1(N-2m)\sigma_3(m)\\
&\qquad +240\cdot 240\sum_{m<\frac{N}{2}}\sigma_3(N-2m)\sigma_3(m)\Bigg\}q^N\\
&= 1 + \sum_{N=1}^{\infty}\Big\{-288N\sigma_1(N) + 240\sigma_3(N) + 240\sigma_3(\frac{N}{2}) - 288\cdot 240N\cdot T_{3,1}(N)\\
&\qquad +288\cdot 240\cdot 2\cdot T_{m,3,1}(N) + 240\cdot 240\cdot T_{3,3}(N)\Big\}q^N,
\end{aligned}
$$

which requests (2.8).

(f) By (3.5) we can know that

$$
\begin{aligned}
&L(q)L(q^8)M(q^2) = L(q)L(q^8)\cdot M(q^2)\\
&= \left(\frac{1}{16}L^2(q) + 4L^2(q^8) - \frac{1}{20}M(q) + \frac{3}{80}M(q^2) + \frac{3}{20}M(q^4) - \frac{16}{5}M(q^8) - 9K(q)\right)\\
&\quad \times M(q^2)\\
&= \frac{1}{16}L^2(q)M(q^2) + 4L^2(q^8)M(q^2) - \frac{1}{20}M(q)M(q^2) + \frac{3}{80}M^2(q^2) + \frac{3}{20}M(q^4)M(q^2)\\
&\quad - \frac{16}{5}M(q^8)M(q^2) - 9K(q)M(q^2)
\end{aligned}
$$

so we refer to (2.6), Proposition 2.2 (e), (i), Proposition 2.3 (j), Theorem 3.2 (b), and (e).

(g) By Proposition 2.2 (a) let us consider Theorem 3.2 (f) in another point of view as

$$
\begin{aligned}
&L(q)L(q^8)M(q^2) = L(q^8)\cdot L(q)M(q^2)\\
&= L(q^8)\left(2L(q^2)M(q^2) + \frac{1}{21}N(q) - \frac{22}{21}N(q^2)\right)\\
&= 2L(q^8)L(q^2)M(q^2) + \frac{1}{21}L(q^8)N(q) - \frac{22}{21}L(q^8)N(q^2)
\end{aligned}
$$

thus we use Proposition 2.2 (j) and Proposition 2.3 (i).

(h) First applying the principle of duplication to (2.2), we have

$$A(q^2) = \frac{x^2\sqrt{1-x}w^6}{256}. \tag{3.6}$$

Thus by (1.10) and (3.6) we can induce that

$$\begin{aligned}
L(q^2)A(q^2) &= \left\{(1-2x)w^2 + 6x(1-x)w\frac{dw}{dx}\right\}\left(\frac{x^2\sqrt{1-x}w^6}{256}\right)\\
&= -\frac{1}{128}x^3\sqrt{1-x}w^8 + \frac{1}{256}x^2\sqrt{1-x}w^8 - \frac{3}{128}x^4\sqrt{1-x}w^7\frac{dw}{dx}\\
&\quad + \frac{3}{128}x^3\sqrt{1-x}w^7\frac{dw}{dx},
\end{aligned}$$

which leads to

$$\begin{aligned}
&-x^4\sqrt{1-x}w^7\frac{dw}{dx} + x^3\sqrt{1-x}w^7\frac{dw}{dx}\\
&= \frac{128}{3}\left(L(q^2)A(q^2) + \frac{1}{128}x^3\sqrt{1-x}w^8 - \frac{1}{256}x^2\sqrt{1-x}w^8\right)\\
&= \frac{128}{3}\left\{L(q^2)A(q^2) + \frac{1}{128}H(q) - \frac{1}{256}\left(8192B(q^4) + 256B(q^2) + \frac{1}{2}H(q)\right)\right\}\\
&= \frac{128}{3}L(q^2)A(q^2) - \frac{4096}{3}B(q^4) - \frac{128}{3}B(q^2) + \frac{1}{4}H(q),
\end{aligned} \tag{3.7}$$

where we use (1.19) and Proposition 2.5 (d). Second from (1.7) and (3.6) we obtain

$$\begin{aligned}
L(q)A(q^2) &= \left\{(1-5x)w^2 + 12x(1-x)w\frac{dw}{dx}\right\}\left(\frac{x^2\sqrt{1-x}w^6}{256}\right)\\
&= -\frac{5}{256}x^3\sqrt{1-x}w^8 + \frac{1}{256}x^2\sqrt{1-x}w^8 - \frac{3}{64}x^4\sqrt{1-x}w^7\frac{dw}{dx}\\
&\quad + \frac{3}{64}x^3\sqrt{1-x}w^7\frac{dw}{dx}\\
&= -\frac{5}{256}x^3\sqrt{1-x}w^8 + \frac{1}{256}x^2\sqrt{1-x}w^8\\
&\quad + \frac{3}{64}\left(-x^4\sqrt{1-x}w^7\frac{dw}{dx} + x^3\sqrt{1-x}w^7\frac{dw}{dx}\right)\\
&= -\frac{5}{256}H(q) + \frac{1}{256}\left(8192B(q^4) + 256B(q^2) + \frac{1}{2}H(q)\right)\\
&\quad + \frac{3}{64}\left(\frac{128}{3}L(q^2)A(q^2) - \frac{4096}{3}B(q^4) - \frac{128}{3}B(q^2) + \frac{1}{4}H(q)\right)\\
&= 2L(q^2)A(q^2) - 32B(q^4) - B(q^2) - \frac{3}{512}H(q),
\end{aligned}$$

where we use (1.19), Proposition 2.5 (d), and (3.7). Finally we refer to Proposition 2.3 (a).

(i) By Proposition 2.2 (d) let us regard Theorem 3.2 (f) as

$$\begin{aligned}
&L(q)L(q^8)M(q^2) = L(q)\cdot L(q^8)M(q^2)\\
&= L(q)\left(\frac{1}{4}L(q^2)M(q^2) - \frac{4}{21}N(q^2) + \frac{5}{28}N(q^4) + \frac{16}{21}N(q^8) + 90A(q^2)\right)\\
&= \frac{1}{4}L(q)L(q^2)M(q^2) - \frac{4}{21}L(q)N(q^2) + \frac{5}{28}L(q)N(q^4) + \frac{16}{21}L(q)N(q^8)\\
&\quad + 90L(q)A(q^2).
\end{aligned}$$

Thus we refer to Proposition 2.2 (f), (h), Proposition 2.3 (e), and Theorem 3.2 (h).

(j) By Proposition 2.2 (b) we have

$$\begin{aligned}
&L(q)L(q^8)M(q^4) = L(q) \cdot L(q^8)M(q^4) \\
&= L(q)\left(\frac{1}{2}L(q^4)M(q^4) - \frac{11}{42}N(q^4) + \frac{16}{21}N(q^8)\right) \\
&= \frac{1}{2}L(q)L(q^4)M(q^4) - \frac{11}{42}L(q)N(q^4) + \frac{16}{21}L(q)N(q^8).
\end{aligned}$$

So we use Proposition 2.2 (h), Proposition 2.3 (k), and Theorem 3.2 (i).

(k) By Proposition 2.2 (c) we can reconsider Theorem 3.2 (j) as

$$\begin{aligned}
&L(q)L(q^8)M(q^4) = L(q^8) \cdot L(q)M(q^4) \\
&= L(q^8)\left(4L(q^4)M(q^4) + \frac{1}{336}N(q) + \frac{5}{112}N(q^2) - \frac{64}{21}N(q^4) - \frac{45}{2}A(q)\right) \\
&= 4L(q^8)L(q^4)M(q^4) + \frac{1}{336}L(q^8)N(q) + \frac{5}{112}L(q^8)N(q^2) - \frac{64}{21}L(q^8)N(q^4) \\
&\quad - \frac{45}{2}L(q^8)A(q),
\end{aligned}$$

which needs Proposition 2.2 (g), (j), Proposition 2.3 (h), and Theorem 3.2 (g).

(l) First by (2.2) and (3.3) we can write $A(q)K(q)$ as

$$\begin{aligned}
A(q)K(q) &= \frac{x(1-x)w^6}{16} \cdot \frac{1}{16}x\sqrt{1-x}w^4 \\
&= \frac{1}{256}x^2\sqrt{1-x}w^{10} - \frac{1}{256}x^3\sqrt{1-x}w^{10}.
\end{aligned} \tag{3.8}$$

Second by (1.11), (3.6), and (3.8) we can note that

$$\begin{aligned}
M(q^2)A(q^2) &= (1-x+x^2)w^4 \cdot \frac{x^2\sqrt{1-x}w^6}{256} \\
&= \frac{1}{256}x^2\sqrt{1-x}w^{10} - \frac{1}{256}x^3\sqrt{1-x}w^{10} + \frac{1}{256}x^4\sqrt{1-x}w^{10} \\
&= A(q)K(q) + \frac{1}{256}x^4\sqrt{1-x}w^{10}.
\end{aligned}$$

Also by (1.14), (3.6), and (3.8) we have

$$\begin{aligned}
M(q^4)A(q^2) &= (1-x+\frac{1}{16}x^2)w^4 \cdot \frac{x^2\sqrt{1-x}w^6}{256} \\
&= \frac{1}{256}x^2\sqrt{1-x}w^{10} - \frac{1}{256}x^3\sqrt{1-x}w^{10} + \frac{1}{16 \cdot 256}x^4\sqrt{1-x}w^{10} \\
&= A(q)K(q) + \frac{1}{16 \cdot 256}x^4\sqrt{1-x}w^{10}.
\end{aligned}$$

Thus the above results lead to

$$M(q^2)A(q^2) - M(q^4)A(q^2) = \frac{15}{4096}x^4\sqrt{1-x}w^{10},$$
$$16M(q^4)A(q^2) - M(q^2)A(q^2) = 15A(q)K(q)$$

and so we have

$$x^4\sqrt{1-x}w^{10} = \frac{4096}{15}M(q^2)A(q^2) - \frac{4096}{15}M(q^4)A(q^2) \tag{3.9}$$

also by Proposition 2.3 (c) and (d) we conclude that

$$A(q)K(q) = \sum_{n=1}^{\infty} c(n)q^{2n}.$$

(m) By (3.3) and (3.6) we obtain

$$A(q^2)K(q) = \frac{x^2\sqrt{1-x}w^6}{256} \cdot \frac{1}{16}x\sqrt{1-x}w^4 = \frac{1}{4096}\left(x^3w^{10} - x^4w^{10}\right)$$

so we refer to Corollary 3.1 (a) and (b).

□

Proof of Lemma 1.1. Now by (3.3) we note that

$$k(n) = 0 \qquad \text{with even } n \in \mathbb{N}. \tag{3.10}$$

And from (1.4) and (3.3) let us consider the following convolution sum :

$$\begin{aligned} 240\sum_{N=1}^{\infty}\left(\sum_{m<\frac{N}{2}}\sigma_3(m)k(N-2m)\right)q^N &= 240\left(\sum_{n=1}^{\infty}k(n)q^n\right)\left(\sum_{m=1}^{\infty}\sigma_3(m)q^{2m}\right) \\ &= K(q)\left(M(q^2)-1\right) \\ &= K(q)M(q^2) - K(q) \end{aligned}$$

then by Theorem 3.2 (b) the above equation constructs

$$\begin{aligned} &\sum_{m<\frac{N}{2}}\sigma_3(m)k(N-2m) \\ &= -\frac{1}{7680}\left\{32k(N) - h(N) - 2g(N) + 512b(\frac{N}{2}) + 16384b(\frac{N}{4})\right\}. \end{aligned} \tag{3.11}$$

If N is even then by (3.10) the left hand side of Eq. (3.11) becomes zero and simultaneously by (1.2) and (3.10) the right hand side of Eq. (3.11) is

$$\sum_{m<\frac{N}{2}}\sigma_3(m)k(N-2m) = \frac{1}{3840}\left\{g(N) - 256b(\frac{N}{2}) - 8192b(\frac{N}{4})\right\}.$$

Therefore we conclude that

$$g(N) - 256b(\frac{N}{2}) - 8192b(\frac{N}{4}) = 0 \quad \text{for even } N.$$

□

4 Proof of Theorem 1.2, Theorem 1.3 and Other Results

Proof of Theorem 1.2. (a) From (1.3) and (1.5) we can observe that

$$\begin{aligned} & 24 \cdot 504 \sum_{N=1}^{\infty} \left(\sum_{m<\frac{N}{8}} \sigma_1(m)\sigma_5(N-8m) \right) q^N \\ & = 24 \cdot 504 \left(\sum_{n=1}^{\infty} \sigma_5(n) q^n \right) \left(\sum_{m=1}^{\infty} \sigma_1(m) q^{8m} \right) = (1-N(q)) \left(1-L(q^8)\right) \\ & = 1 - L(q^8) - N(q) + N(q)L(q^8) \end{aligned}$$

thus we use Theorem 3.2 (g) and we have

$$\begin{aligned} & \sum_{m<\frac{N}{8}} \sigma_1(m)\sigma_5(N-8m) \\ & = \frac{1}{2193408} \Big\{ 1344\sigma_7(N) + 4032\sigma_7(\frac{N}{2}) + 16128\sigma_7(\frac{N}{4}) + 65536\sigma_7(\frac{N}{8}) \\ & - 22848\,(N-4)\,\sigma_5(N) + 4352\sigma_1(\frac{N}{8}) - 1071h(N) - 2142g(N) - 35616b(N) \\ & -1421952b(\frac{N}{2}) - 18665472b(\frac{N}{4}) \Big\}. \end{aligned} \tag{4.1}$$

If N is odd then it is obvious that

$$\begin{aligned} & \sum_{m<\frac{N}{8}} \sigma_1(m)\sigma_5(N-8m) \\ & = \frac{1}{104448} \left\{ 64\sigma_7(N) - 1088\,(N-4)\,\sigma_5(N) - 51h(N) - 102g(N) - 1696b(N) \right\} \end{aligned}$$

but if N is even then we apply (1.2) and Lemma 1.1 to Eq. (4.1).

(b) By (1.3) and (1.5) we can know that

$$\begin{aligned} & 24 \cdot 504 \sum_{N=1}^{\infty} \left(\sum_{m<\frac{N}{8}} \sigma_5(m)\sigma_1(N-8m) \right) q^N \\ & = 24 \cdot 504 \left(\sum_{n=1}^{\infty} \sigma_1(n) q^n \right) \left(\sum_{m=1}^{\infty} \sigma_5(m) q^{8m} \right) = (1-L(q)) \left(1-N(q^8)\right) \\ & = 1 - N(q^8) - L(q) + L(q)N(q^8), \end{aligned}$$

which requests Theorem 3.2 (i) then we obtain

$$\sum_{m<\frac{N}{8}} \sigma_5(m)\sigma_1(N-8m)$$
$$= \frac{1}{280756224}\Big\{32\sigma_7(N) + 2016\sigma_7(\frac{N}{2}) + 129024\sigma_7(\frac{N}{4}) + 11010048\sigma_7(\frac{N}{8})$$
$$- 11698176\,(2N-1)\,\sigma_5(\frac{N}{8}) + 557056\sigma_1(N) + 7497h(N) - 17136g(N)$$
$$-282912b(N) + 446208b(\frac{N}{2}) + 122142720b(\frac{N}{4})\Big\}. \tag{4.2}$$

If N is odd then it is exact that

$$\sum_{m<\frac{N}{8}} \sigma_5(m)\sigma_1(N-8m)$$
$$= \frac{1}{280756224}\left\{32\sigma_7(N) + 557056\sigma_1(N) + 7497h(N) - 17136g(N) - 282912b(N)\right\}$$

otherwise if N is even then we apply (1.2) and Lemma 1.1 to Eq. (4.2).

□

Proposition 4.1. *(See ([15], Theorem 3.1(ii))) For a prime p and $s, n \in \mathbb{N}$ we have*

$$\sigma_s(pn) - (p^s+1)\,\sigma_s(n) + p^s\sigma_s(\frac{n}{p}) = 0.$$

Corollary 4.1. *Let $n \in \mathbb{N}$. Then we have*

(a)

$$\sum_{m<\frac{n}{2}} a(m)\sigma_1(n-2m) = \frac{1}{24}\left\{\frac{3}{512}h(n) + 3b(\frac{n}{2}) + 96b(\frac{n}{4}) - (2n-1)\,a(\frac{n}{2})\right\},$$

(b)

$$\sum_{m<\frac{n}{8}} \sigma_1(m)a(n-8m)$$
$$= \begin{cases} 0, & \textit{for even } n, \\ \frac{1}{6144}\left\{3h(n) - 6g(n) - 96b(n) - 64\,(n-4)\,a(n)\right\}, & \textit{for odd } n, \end{cases}$$

(c)

$$\sum_{m=1}^{n-1} g(2m)\sigma_1(n-m) = \begin{cases} 128\left\{6d(n) - (3n-2)\,b(\frac{n}{2})\right\}, & \textit{for even } n, \\ \frac{16}{3}\left\{d(2n) - 192d(n) - (3n-2)\,b(n)\right\}, & \textit{for odd } n, \end{cases}$$

(d)

$$\sum_{m=1}^{n-1} g(2m)\sigma_1(2n-2m) = \begin{cases} 256\left\{8d(n) - (3n-1)\,b(\frac{n}{2})\right\}, & \textit{for even } n, \\ \frac{32}{3}\left\{2d(2n) - 288d(n) - (3n-1)\,b(n)\right\}, & \textit{for odd } n, \end{cases}$$

(e)

$$\sum_{m<\frac{n+1}{2}} g(4m-2)\sigma_1(n-2m+1)$$
$$= \begin{cases} 256d(n), & \textit{for even } n, \\ \frac{16}{3}\left\{d(2n) - 48d(n) - (3n-2)\,b(n)\right\}, & \textit{for odd } n, \end{cases}$$

(f)

$$\sum_{m=1}^{n-1} g(2m)\sigma_3(n-m)$$
$$= \begin{cases} \frac{128}{5}\left\{11\tau(\frac{n}{2}) + 2816\tau(\frac{n}{4}) - b(\frac{n}{2})\right\}, & \textit{for even } n, \\ \frac{16}{10365}\left\{30\sigma_{11}(n) + 661\tau(n) - 1358561280f(n) - 691b(n)\right\}, & \textit{for odd } n, \end{cases}$$

(g)

$$\sum_{m=1}^{n-1} g(2m)\sigma_3(2n-2m)$$
$$= \begin{cases} \frac{128}{5}\left\{91\tau(\frac{n}{2}) + 23296\tau(\frac{n}{4}) - b(\frac{n}{2})\right\}, & \textit{for even } n, \\ \frac{16}{10365}\left\{270\sigma_{11}(n) + 421\tau(n) - 12227051520f(n) - 691b(n)\right\}, & \textit{for odd } n, \end{cases}$$

(h)

$$\sum_{m<\frac{n+1}{2}} g(4m-2)\sigma_3(n-2m+1)$$
$$= \begin{cases} 256\left\{\tau(\frac{n}{2}) + 64\tau(\frac{n}{4})\right\}, & \textit{for even } n, \\ \frac{8}{10365}\left\{15\sigma_{11}(n) + 1367\tau(n) - 679280640f(n) - 1382b(n)\right\}, & \textit{for odd } n. \end{cases}$$

Proof. (a) By (1.3) and (2.1) we note that

$$\begin{aligned} 24\sum_{N=1}^{\infty}\left(\sum_{m<\frac{N}{2}} a(m)\sigma_1(N-2m)\right)q^N &= 24\left(\sum_{n=1}^{\infty}\sigma_1(n)q^n\right)\left(\sum_{m=1}^{\infty}a(m)q^{2m}\right) \\ &= (1-L(q))\,A(q^2) \\ &= A(q^2) - L(q)A(q^2). \end{aligned}$$

So we refer to Theorem 3.2 (h).

(b) By (1.3) and (2.1) we can see that

$$24\sum_{N=1}^{\infty}\left(\sum_{m<\frac{N}{8}}\sigma_1(m)a(N-8m)\right)q^N = 24\left(\sum_{n=1}^{\infty}a(n)q^n\right)\left(\sum_{m=1}^{\infty}\sigma_1(m)q^{8m}\right)$$
$$= A(q)\left(1-L(q^8)\right)$$
$$= A(q) - A(q)L(q^8).$$

Therefore using Theorem 3.2 (k) we obtain

$$\sum_{m<\frac{N}{8}}\sigma_1(m)a(N-8m)$$
$$= \frac{1}{6144}\left\{3h(N)-6g(N)-96b(N)+768b(\frac{N}{2})+49152b(\frac{N}{4})-64\,(N-4)\,a(N)\right\},$$

which shows that for odd N since

$$b(\frac{N}{2}) = b(\frac{N}{4}) = 0,$$

thus we can easily have

$$\sum_{m<\frac{N}{8}}\sigma_1(m)a(N-8m) = \frac{1}{6144}\left\{3h(N)-6g(N)-96b(N)-64\,(N-4)\,a(N)\right\}$$

but for even N it is definite

$$\sum_{m<\frac{N}{8}}\sigma_1(m)a(N-8m) = 0$$

by (2.4).

(c) In Lemma 1.1 we can replace n with $2m$ for $m \in \mathbb{N}$ thus

$$g(2m) = 256b(m) + 8192b(\frac{m}{2}) \tag{4.3}$$

and so we can write

$$\sum_{m=1}^{n-1}g(2m)\sigma_1(n-m)$$
$$= \sum_{m=1}^{n-1}\left\{256b(m)+8192b(\frac{m}{2})\right\}\sigma_1(n-m)$$
$$= 256\sum_{m=1}^{n-1}b(m)\sigma_1(n-m) + 8192\sum_{m<\frac{n}{2}}b(m)\sigma_1(n-2m).$$

Therefore we use Proposition 2.4 (a) and (b) to obtain

$$\sum_{m=1}^{n-1} g(2m)\sigma_1(n-m) = -\frac{16}{3}\Big\{160d(n) - 10240d(\frac{n}{2}) - c(n) - 320c(\frac{n}{2}) + (3n-2)\, b(n) + 32\,(3n-2)\, b(\frac{n}{2})\Big\}. \tag{4.4}$$

Applying (2.3) to the above equation we have

$$\sum_{m=1}^{n-1} g(2m)\sigma_1(n-m) = \frac{16}{3}\Big\{d(2n) - 192d(n) + 10240d(\frac{n}{2}) + 320c(\frac{n}{2}) - (3n-2)\, b(n) - 32\,(3n-2)\, b(\frac{n}{2})\Big\}. \tag{4.5}$$

So if n is odd then it is clear that

$$\sum_{m=1}^{n-1} g(2m)\sigma_1(n-m) = \frac{16}{3}\{d(2n) - 192d(n) - (3n-2)\, b(n)\}$$

but if n is even then by (1.2) and (2.4), and Proposition 2.1 (a), Eq. (4.5) becomes

$$\sum_{m=1}^{n-1} g(2m)\sigma_1(n-m) = 128\Big\{6d(n) - (3n-2)\, b(\frac{n}{2})\Big\}.$$

(d) By Proposition 4.1 and (4.3) we have

$$\begin{aligned}
&\sum_{m=1}^{n-1} g(2m)\sigma_1(2n-2m)\\
&= \sum_{m=1}^{n-1}\Big\{256b(m) + 8192b(\frac{m}{2})\Big\}\Big\{3\sigma_1(n-m) - 2\sigma_1(\frac{n-m}{2})\Big\}\\
&= 256\cdot 3\sum_{m=1}^{n-1} b(m)\sigma_1(n-m) - 256\cdot 2\sum_{m<\frac{n}{2}} b(n-2m)\sigma_1(m)\\
&\quad + 8192\cdot 3\sum_{m<\frac{n}{2}} b(m)\sigma_1(n-2m) - 8192\cdot 2\sum_{m<\frac{n}{2}} b(m)\sigma_1(\frac{n}{2}-m)
\end{aligned}$$

which request Proposition 2.4 (a), (b), and (c). Then we obtain

$$\sum_{m=1}^{n-1} g(2m)\sigma_1(2n-2m) = -\frac{32}{3}\Big\{224d(n) - 14336d(\frac{n}{2}) - 2c(n) - 448c(\frac{n}{2}) + (3n-1)\, b(n) + 32\,(3n-1)\, b(\frac{n}{2})\Big\}. \tag{4.6}$$

Applying (2.3) to Eq. (4.6) we have

$$\sum_{m=1}^{n-1} g(2m)\sigma_1(2n-2m) = \frac{32}{3}\Big\{2d(2n) - 288d(n) + 14336d(\frac{n}{2}) + 448c(\frac{n}{2}) - (3n-1)\,b(n) - 32\,(3n-1)\,b(\frac{n}{2})\Big\}. \tag{4.7}$$

So if n is odd then it is obvious that

$$\sum_{m=1}^{n-1} g(2m)\sigma_1(2n-2m) = \frac{32}{3}\left\{2d(2n) - 288d(n) - (3n-1)\,b(n)\right\}$$

but if n is even then we apply (1.2), (2.4), and Proposition 2.1 (a) to Eq. (4.7).

(e) By (1.2) and Lemma 1.1, let us expand Corollary 4.1 (c) as

$$\begin{aligned}
&\sum_{m=1}^{n-1} g(2m)\sigma_1(n-m) \\
&= \sum_{m<\frac{n}{2}} g(4m)\sigma_1(n-2m) + \sum_{m<\frac{n+1}{2}} g(4m-2)\sigma_1(n-2m+1) \\
&= \sum_{m<\frac{n}{2}} \left\{256b(2m) + 8192b(m)\right\}\sigma_1(n-2m) + \sum_{m<\frac{n+1}{2}} g(4m-2)\sigma_1(n-2m+1) \\
&= \sum_{m<\frac{n}{2}} \left\{256(-8b(m)) + 8192b(m)\right\}\sigma_1(n-2m) + \sum_{m<\frac{n+1}{2}} g(4m-2)\sigma_1(n-2m+1) \\
&= 6144\sum_{m<\frac{n}{2}} b(m)\sigma_1(n-2m) + \sum_{m<\frac{n+1}{2}} g(4m-2)\sigma_1(n-2m+1)
\end{aligned}$$

so we refer to Proposition 2.4 (a) and we have

$$\sum_{m<\frac{n+1}{2}} g(4m-2)\sigma_1(n-2m+1) = -\frac{16}{3}\Big\{16d(n) - 2560d(\frac{n}{2}) - c(n) - 80c(\frac{n}{2}) + (3n-2)\,b(n) + 8\,(3n-2)\,b(\frac{n}{2})\Big\}. \tag{4.8}$$

Applying (2.3) to the above identity we obtain

$$\sum_{m<\frac{n+1}{2}} g(4m-2)\sigma_1(n-2m+1) = \frac{16}{3}\Big\{d(2n) - 48d(n) + 2560d(\frac{n}{2}) + 80c(\frac{n}{2}) - (3n-2)\,b(n) - 8\,(3n-2)\,b(\frac{n}{2})\Big\}. \tag{4.9}$$

Therefore if n is odd then we use

$$d(\frac{n}{2}) = c(\frac{n}{2}) = b(\frac{n}{2}) = 0$$

to Eq. (4.9) otherwise if n is even then we utilize (1.2), (2.4), and Proposition 2.1 (a) to the same equation.

(f) By (4.3) we note that

$$\begin{aligned}&\sum_{m=1}^{n-1} g(2m)\sigma_3(n-m)\\&=\sum_{m=1}^{n-1}\left\{256b(m)+8192b(\frac{m}{2})\right\}\sigma_3(n-m)\\&=256\sum_{m=1}^{n-1} b(m)\sigma_3(n-m)+8192\sum_{m<\frac{n}{2}} b(m)\sigma_3(n-2m)\end{aligned}$$

thus appealing to Proposition 2.4 (d) and (f) we obtain

$$\begin{aligned}\sum_{m=1}^{n-1} g(2m)\sigma_3(n-m)=\frac{16}{10365}&\left\{30\sigma_{11}(n)-30\sigma_{11}(\frac{n}{2})+661\tau(n)+136848\tau(\frac{n}{2})\right.\\&-79249408\tau(\frac{n}{4})-1358561280f(n)-691b(n)\\&\left.-22112b(\frac{n}{2})\right\}.\end{aligned}\tag{4.10}$$

So if n is odd then (4.10) is changed as

$$\sum_{m=1}^{n-1} g(2m)\sigma_3(n-m)=\frac{16}{10365}\left\{30\sigma_{11}(n)+661\tau(n)-1358561280f(n)-691b(n)\right\}$$

but if n is even then we use (1.2), Proposition 2.1 (b), and (c) to (4.10).

(g) From Proposition 4.1 and (4.3) we expand

$$\begin{aligned}&\sum_{m=1}^{n-1} g(2m)\sigma_3(2n-2m)\\&=\sum_{m=1}^{n-1}\left\{256b(m)+8192b(\frac{m}{2})\right\}\left\{9\sigma_3(n-m)-8\sigma_3(\frac{n-m}{2})\right\}\\&=256\cdot 9\sum_{m=1}^{n-1} b(m)\sigma_3(n-m)-256\cdot 8\sum_{m<\frac{n}{2}} b(n-2m)\sigma_3(m)\\&\quad+8192\cdot 9\sum_{m<\frac{n}{2}} b(m)\sigma_3(n-2m)-8192\cdot 8\sum_{m<\frac{n}{2}} b(m)\sigma_3(\frac{n}{2}-m)\end{aligned}$$

therefore we refer to Proposition 2.4 (d), (e), and (f) to have

$$\begin{aligned}\sum_{m=1}^{n-1} g(2m)\sigma_3(2n-2m)=\frac{16}{10365}&\left\{270\sigma_{11}(n)-270\sigma_{11}(\frac{n}{2})+421\tau(n)+966288\tau(\frac{n}{2})\right.\\&-758530048\tau(\frac{n}{4})-12227051520f(n)-691b(n)\\&\left.-22112b(\frac{n}{2})\right\}.\end{aligned}\tag{4.11}$$

So for odd n, the convolution sum formula $\sum_{m=1}^{n-1} g(2m)\sigma_3(2n-2m)$ is calculated easily but for even n we use (1.2), Proposition 2.1 (b), and (c) to (4.11).

(h) From proof of Corollary 4.1 (e) we can induce that

$$\begin{aligned}&\sum_{m<\frac{n+1}{2}} g(4m-2)\sigma_3(n-2m+1)\\&=\sum_{m=1}^{n-1} g(2m)\sigma_3(n-m) - 6144\sum_{m<\frac{n}{2}} b(m)\sigma_3(n-2m).\end{aligned}$$

So we need Proposition 2.4 (f) and Corollary 4.1 (f) to obtain

$$\begin{aligned}&\sum_{m<\frac{n+1}{2}} g(4m-2)\sigma_3(n-2m+1)\\&=\frac{8}{10365}\Big\{15\sigma_{11}(n)-15\sigma_{11}(\frac{n}{2})+1367\tau(n)+333768\tau(\frac{n}{2})-39624704\tau(\frac{n}{4})\\&\quad-679280640f(n)-1382b(n)-11056b(\frac{n}{2})\Big\}.\end{aligned} \tag{4.12}$$

Thus for only even n we apply (1.2), Proposition 2.1 (b), and (c) to (4.12).

□

Remark 4.1. By (4.3) let us expand Corollary 4.1 (e) as

$$\begin{aligned}&\sum_{m<\frac{n+1}{2}} g(4m-2)\sigma_1(n-2m+1)\\&=\sum_{m<\frac{n+1}{2}} g(2(2m-1))\sigma_1(n-2m+1)\\&=\sum_{m<\frac{n+1}{2}}\left\{256b(2m-1)+8192b(\frac{2m-1}{2})\right\}\sigma_1(n-2m+1)\\&=256\sum_{m<\frac{n+1}{2}} b(2m-1)\sigma_1(n-2m+1)\end{aligned}$$

thus we obtain

$$\begin{aligned}&\sum_{m<\frac{n+1}{2}} b(2m-1)\sigma_1(n-2m+1)\\&=\begin{cases} d(n), & \text{for even } n,\\ \frac{1}{48}\left\{d(2n)-48d(n)-(3n-2)\,b(n)\right\}, & \text{for odd } n.\end{cases}\end{aligned}$$

On the other hand, by (4.8) we can also have

$$\sum_{m<\frac{n+1}{2}} b(2m-1)\sigma_1(n-2m+1) = -\frac{1}{48}\Big\{16d(n) - 2560d(\frac{n}{2}) - c(n) - 80c(\frac{n}{2}) + (3n-2)\,b(n) + 8\,(3n-2)\,b(\frac{n}{2})\Big\}.$$

Similarly by Corollary 4.1 (h) we obtain

$$\sum_{m<\frac{n+1}{2}} b(2m-1)\sigma_3(n-2m+1) = \begin{cases} \tau(\frac{n}{2}) + 64\tau(\frac{n}{4}), & \text{for even } n, \\ \frac{1}{331680}\left\{15\sigma_{11}(n) + 1367\tau(n) - 679280640f(n) - 1382b(n)\right\}, & \text{for odd } n, \end{cases}$$

and by (4.12) we conclude that

$$\sum_{m<\frac{n+1}{2}} b(2m-1)\sigma_3(n-2m+1) = \frac{1}{331680}\Big\{15\sigma_{11}(n) - 15\sigma_{11}(\frac{n}{2}) + 1367\tau(n) + 333768\tau(\frac{n}{2}) - 39624704\tau(\frac{n}{4}) - 679280640f(n) - 1382b(n) - 11056b(\frac{n}{2})\Big\}.$$

Proof of Theorem 1.3. In advance, by Lemma 1.1 let us investigate the property of $g(2^k m)$:
If $k=2$ then by (1.2) we have

$$g(2^2m) = g(4m) = 256b(2m) + 8192b(m) = 256\,(-8b(m)) + 8192b(m) = (-1)^2\cdot 3\cdot 2^{11}b(m).$$

And if $k=3$ then by (1.2) and the above identity of $g(2^2m)$, we have

$$g(2^3m) = g(2^2(2m)) = (-1)^2\cdot 3\cdot 2^{11}b(2m) = (-1)^2\cdot 3\cdot 2^{11}\,(-8b(m)) = (-1)^3\cdot 3\cdot 2^{11+3}b(m).$$

Similarly if $k=4$ then by (1.2) and $g(2^3m)$, we obtain

$$g(2^4m) = g(2^3(2m)) = (-1)^3\cdot 3\cdot 2^{11+3}b(2m) = (-1)^3\cdot 3\cdot 2^{11+3}\,(-8b(m)) = (-1)^4\cdot 3\cdot 2^{11+3+3}b(m).$$

Continuing this process we conclude that

$$g(2^k m) = 3(-1)^k\cdot 2^{11+3(k-2)}b(m) = 96(-2)^{3k}b(m). \tag{4.13}$$

(a) By (4.13) we can observe that

$$\sum_{m=1}^{n-1} g(2^k m)\sigma_1(n-m) = \sum_{m=1}^{n-1} 96(-2)^{3k}b(m)\sigma_1(n-m) = 96(-2)^{3k}\sum_{m=1}^{n-1} b(m)\sigma_1(n-m)$$

so we use Proposition 2.4 (b) and have

$$\sum_{m=1}^{n-1} g(2^k m)\sigma_1(n-m) = (-1)^{3k} \cdot 2^{3k+1} \{32d(n) + c(n) - (3n-2)\, b(n)\} .$$

Finally we appeal to (2.3).

(b) By (4.13) we can easily know that

$$\begin{aligned}\sum_{m<\frac{n}{2}} g(2^k m)\sigma_1(n-2m) &= \sum_{m<\frac{n}{2}} 96(-2)^{3k} b(m)\sigma_1(n-2m)\\ &= 96(-2)^{3k} \sum_{m<\frac{n}{2}} b(m)\sigma_1(n-2m),\end{aligned}$$

which requests Proposition 2.4 (a) thus we obtain

$$\begin{aligned}&\sum_{m<\frac{n}{2}} g(2^k m)\sigma_1(n-2m)\\ &= (-2)^{3k+1} \left\{6d(n) - 320d(\frac{n}{2}) - 10c(\frac{n}{2}) + (3n-2)\, b(\frac{n}{2})\right\} .\end{aligned} \tag{4.14}$$

So if n is even then we apply (2.4) to Eq. (4.14) but for odd n it is obvious.

(c) Let us consider Theorem 1.3 (a) in another point of view as

$$\begin{aligned}&\sum_{m=1}^{n-1} g(2^k m)\sigma_1(n-m)\\ &= \sum_{m<\frac{n}{2}} g(2^k \cdot 2m)\sigma_1(n-2m) + \sum_{m<\frac{n+1}{2}} g(2^k(2m-1))\sigma_1(n-2m+1)\\ &= \sum_{m<\frac{n}{2}} g(2^{k+1} m)\sigma_1(n-2m) + \sum_{m<\frac{n+1}{2}} g(2^k(2m-1))\sigma_1(n-2m+1)\end{aligned}$$

thus we replace k with $k+1$ in (4.14) and insert the obtained value in the above equation so that we have

$$\begin{aligned}&\sum_{m<\frac{n+1}{2}} g(2^k(2m-1))\sigma_1(n-2m+1)\\ &= -(-2)^{3k+1} \left\{d(2n) - 48d(n) + 2560d(\frac{n}{2}) + 80c(\frac{n}{2}) - (3n-2)\, b(n)\right.\\ &\qquad \left. -8\,(3n-2)\, b(\frac{n}{2})\right\} .\end{aligned} \tag{4.15}$$

If n is even then we use (1.2), (2.4), and Proposition 2.1 (a) in (4.15) but for odd n we can easily simplify (4.15).

(d) Proof is similar manner to proof of Theorem 1.3 (a) except we refer to Proposition 2.4 (d).

(e) We follow proof of Theorem 1.3 (b) but we need Proposition 2.4 (f) and especially for even n, we use Proposition 2.1 (b) and (c).

(f) In similar manner to proof of Theorem 1.3 (c) we proceed and especially for even n, we should refer to (1.2), Proposition 2.1 (b) and (c).

□

5 Conclusions

In this paper, mainly we are focused on the convolution sum formulae as

$$\sum_{m<\frac{n}{8}} \sigma_1(m)\sigma_5(n-8m) \quad \text{and} \quad \sum_{m<\frac{n}{8}} \sigma_5(m)\sigma_1(n-8m)$$

where $n \in \mathbb{N}$. And collaterally, we construct new convolution sums with the coefficients $b(n)$, $g(n)$, and divisor functions and deduce some formulae.

Competing Interests

The author declares that no competing interests exist.

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Appendix

The first eighteen values of $\tau(n)$ are given in the Table 1,

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
1	1	7	−16744	13	−577738
2	−24	8	84480	14	401856
3	252	9	−113643	15	1217160
4	−1472	10	−115920	16	987136
5	4830	11	534612	17	−6905934
6	−6048	12	−370944	18	2727432

TABLE 1. $\tau(n)$ for n $(1 \leq n \leq 18)$

similarly the first eighteen values of $a(n)$, $b(n)$, $c(n)$, $d(n)$, $f(n)$, $g(n)$, $h(n)$, and $k(n)$ are listed in the following tables.

n	$a(n)$	n	$a(n)$	n	$a(n)$
1	1	7	−88	13	−418
2	0	8	0	14	0
3	−12	9	−99	15	−648
4	0	10	0	16	0
5	54	11	540	17	594
6	0	12	0	18	0

TABLE 2. $a(n)$ for n $(1 \leq n \leq 18)$

n	$b(n)$	n	$b(n)$	n	$b(n)$
1	1	7	1016	13	1382
2	−8	8	−512	14	−8128
3	12	9	−2043	15	−2520
4	64	10	1680	16	4096
5	−210	11	1092	17	14706
6	−96	12	768	18	16344

TABLE 3. $b(n)$ for n $(1 \leq n \leq 18)$

n	$c(n)$	n	$c(n)$	n	$c(n)$
1	1	7	−4536	13	37806
2	−16	8	−4096	14	15232
3	100	9	23085	15	−146472
4	−256	10	−13920	16	−65536
5	−154	11	−38996	17	311442
6	2496	12	39936	18	−74448

TABLE 4. $c(n)$ for n $(1 \leq n \leq 18)$

n	$d(n)$	n	$d(n)$	n	$d(n)$
1	0	7	112	13	4384
2	1	8	256	14	−952
3	−8	9	−576	15	336
4	16	10	870	16	4096
5	32	11	−536	17	−17472
6	−156	12	−2496	18	4653

TABLE 5. $d(n)$ for n $(1 \leq n \leq 18)$

n	$f(n)$	n	$f(n)$	n	$f(n)$
1	0	7	44	13	39569
2	0	8	192	14	89424
3	0	9	694	15	191028
4	0	10	2208	16	388608
5	1	11	6296	17	756822
6	8	12	16384	18	1419200

TABLE 6. $f(n)$ for n $(1 \leq n \leq 18)$

n	$g(n)$	n	$g(n)$	n	$g(n)$
1	16	7	−25728	13	233056
2	256	8	−49152	14	260096
3	1728	9	−44976	15	398976
4	6144	10	−53760	16	393216
5	10976	11	−55744	17	−301280
6	3072	12	73728	18	−523008

TABLE 7. $g(n)$ for n $(1 \leq n \leq 18)$

n	$h(n)$	n	$h(n)$	n	$h(n)$
1	0	7	−24576	13	540672
2	0	8	0	14	0
3	4096	9	−163840	15	385024
4	0	10	0	16	0
5	16384	11	−20480	17	−163840
6	0	12	0	18	0

TABLE 8. $h(n)$ for n $(1 \leq n \leq 18)$

n	$k(n)$	n	$k(n)$	n	$k(n)$
1	1	7	24	13	22
2	0	8	0	14	0
3	-4	9	-11	15	8
4	0	10	0	16	0
5	-2	11	-44	17	50
6	0	12	0	18	0

TABLE 9. $k(n)$ for n $(1 \leq n \leq 18)$

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